Grade ⇒ 1-2-3 (45) 4-5-6 (45) 15 July 2019 / 18 July 2019

-15 30 60 7 -15 30 ชั่วโมง

นิยามอินทิกรัล $\iint_R f\,dA$ $\Delta x = \frac{b-a}{n}$, $\Delta y = \frac{d-c}{m}$, $x_i = a+i\Delta x$, $y_j = c+j\Delta y$

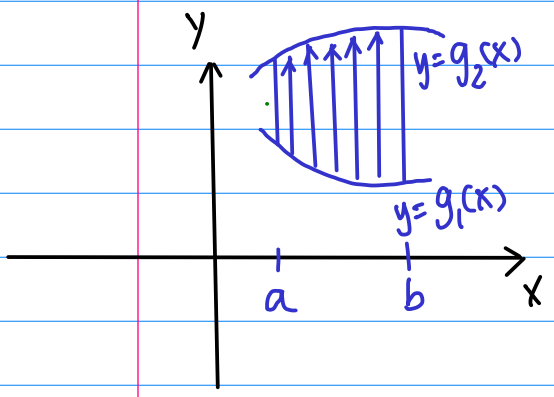
1. $\iint_{R=[a,b]\times[c,d]} f\,dA = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\,\Delta x\,\Delta y$, $(x_{ij}^*, y_{ij}^*) \in [x_{i-1}, x_i]\times[y_{j-1}, y_j]$

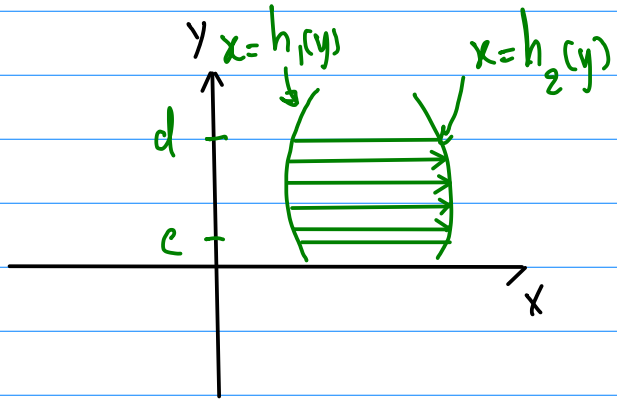
2. $\iint_R f\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$

3. $\iint_R f\pm g\,dA$, $\iint_R cf\,dA$

4. $\iint_D f\,dA \rightarrow D = \{(x,y) \mid g_1(x) \le y \le g_2(x),\ a \le x \le b\}$ (ต้องหา, อินทิเกรตหลอกก่อน)

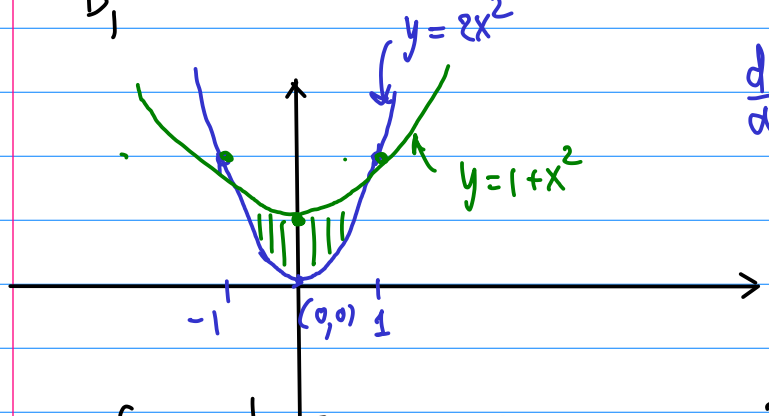
$D = \{(x,y) \mid h_1(y) \le x \le h_2(y),\ c \le y \le d\}$





โจทย์ $\iint_{D_1}(2x+y)\,dA$ โดยที่ D_1 อยู่ในบริเวณปิดระหว่าง $y=2x^2$, $y=1+x^2$

วิธีทำ



$y=2x^2$ แทน $x=0 \Rightarrow y=0$

$\frac{dy}{dx} = 4x$ $x=\pm1 \Rightarrow y=2$

$y=1+x^2$ แทน $x=0 \Rightarrow y=1$

$x=\pm1 \Rightarrow y=2$

$\frac{dy}{dx} = 2x$

$D_1 = \{(x,y) \mid 2x^2 \le y \le 1+x^2,\ -1 \le x \le 1\}$

กรอบใน (ก่อน) กรอบนอก

$-\frac{3}{2}x^4 - 2x^3 + x^2 + 2x + \frac{1}{2}$ $\qquad 1+x^2-2x^2 = 1-x^2$ $\qquad 1+2x^2+x^4-4x^4$

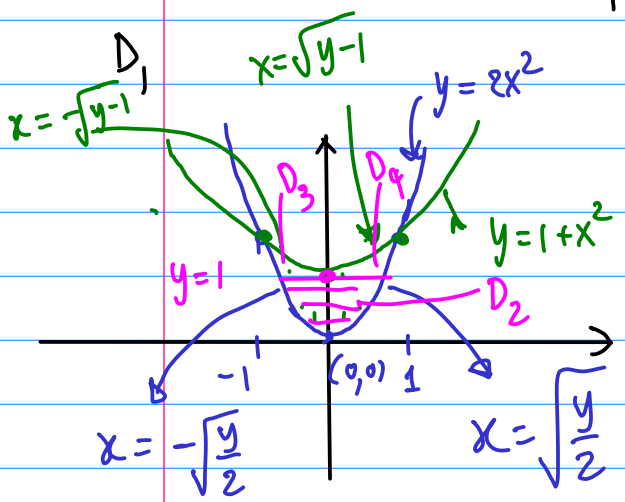
$$\iint_{D_1}(2x+y)\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2}(2x+y)\,dy\,dx = \int_{-1}^{1} 2xy\Big|_{y=2x^2}^{y=1+x^2} + \frac{y^2}{2}\Big|_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} 2x - 2x^3 + \frac{1}{2}\left(1+2x^2-3x^4\right)dx \qquad \int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$= x^2\Big|_{x=-1}^{x=1} - \frac{2}{4}x^4\Big|_{x=-1}^{x=1} + \frac{1}{2}x\Big|_{x=-1}^{x=1} + \frac{x^3}{3}\Big|_{x=-1}^{x=1} - \frac{3}{2}\frac{x^5}{5}\Big|_{x=-1}^{x=1}$$

$$= 1-(-1)^2 - \frac{1}{2}\left(1^4-(-1)^4\right) + \frac{1}{2}(1-(-1)) + \frac{1}{3}\left(1^3-(-1)^3\right) - \frac{3}{10}\left(1^5-(-1)^5\right)$$

$$= 1 + \frac{2}{3} - \frac{3}{5} = \frac{15+10-9}{15} = \frac{16}{15}$$



$$D_2 = \left\{(x,y)\,\middle|\, -\sqrt{\tfrac{y}{2}} \le x \le \sqrt{\tfrac{y}{2}},\ 0 \le y \le 1\right\}$$

$$D_3 = \left\{(x,y)\,\middle|\, -\sqrt{\tfrac{y}{2}} \le x \le -\sqrt{y-1},\ 1 \le y \le 2\right\}$$

$$D_4 = \left\{(x,y)\,\middle|\, \sqrt{y-1} \le x \le \sqrt{\tfrac{y}{2}},\ 1 \le y \le 2\right\}$$

$$D_1 = D_2 \cup D_3 \cup D_4$$

$$\iint_{D_1}(2x+y)\,dA = \iint_{D_2}(2x+y)\,dA + \iint_{D_3}(2x+y)\,dA + \iint_{D_4}(2x+y)\,dA$$

$$\iint_{D_2}(2x+y)\,dA = \int_0^1\int_{-\sqrt{\frac{y}{2}}}^{\sqrt{\frac{y}{2}}}(2x+y)\,dx\,dy = \int_0^1 x^2\Big|_{x=-\sqrt{\frac{y}{2}}}^{x=\sqrt{\frac{y}{2}}} + yx\Big|_{x=-\sqrt{\frac{y}{2}}}^{x=\sqrt{\frac{y}{2}}} dy = \int_0^1 \frac{y}{2} - \left(-\sqrt{\frac{y}{2}}\right)^2 + y\left(\sqrt{\frac{y}{2}} + \sqrt{\frac{y}{2}}\right)dy$$

$$= \sqrt{2}\int_0^1 y^{\frac{3}{2}}\,dy = 2\sqrt{2}\,\frac{y^{\frac{5}{2}}}{5}\Big|_{y=0}^{y=1} = \frac{2\sqrt{2}}{5}$$

$-\frac{\sqrt{2}}{5}+\frac{17}{60}$

$$\iint_{D_3}(2x+y)dA=\int_1^2\int_{-\sqrt{\frac{y}{2}}}^{-\sqrt{y-1}}(2x+y)\,dx\,dy=\int_1^2 x^2\Big|_{x=-\sqrt{\frac{y}{2}}}^{x=-\sqrt{y-1}}+yx\Big|_{x=-\sqrt{\frac{y}{2}}}^{x=-\sqrt{y-1}}dy=\int_1^2 y-1-\frac{y}{2}-y\sqrt{y-1}+\frac{y^{\frac{3}{2}}}{\sqrt{2}}\,dy$$

no.
$$\int y\sqrt{y-1}\,dy=\int(u+1)\sqrt{u}\,du=\int u^{\frac{3}{2}}du+\int u^{\frac{1}{2}}du=\frac{2u^{\frac{5}{2}}}{5}+\frac{2u^{\frac{3}{2}}}{3}+C$$

(subst. $u=y-1$, $du=dy$)

$4\sqrt{2}$

$$\iint_{D_3}(2x+y)dA=\frac{y^2}{2}\Big|_{y=1}^{y=2}-y\Big|_{y=1}^{y=2}-\frac{y^2}{4}\Big|_{y=1}^{y=2}-\frac{2}{5}(y-1)^{\frac{5}{2}}\Big|_{y=1}^{y=2}-\frac{2}{3}(y-1)^{\frac{3}{2}}\Big|_{y=1}^{y=2}+\frac{2}{\sqrt{2}}\frac{y^{\frac{5}{2}}}{5}\Big|_{y=1}^{y=2}$$

$\frac{1}{2}$ $\qquad$ $\frac{6}{5}$ $\qquad$ $102-85=17$

$$=\frac{3}{2}-\frac{1}{\sqrt{2}}-\frac{3}{4}-\frac{2}{5}-\frac{2}{3}+\frac{8}{5}-\frac{\sqrt{2}}{5}=\frac{30-45-40+72-12\sqrt{2}}{60}=\frac{17-12\sqrt{2}}{60}\ \checkmark$$

$-\frac{\sqrt{2}}{5}+\frac{47}{60}$

$$\iint_{D_4}(2x+y)dA=\int_1^2\int_{\sqrt{y-1}}^{\sqrt{\frac{y}{2}}}(2x+y)\,dx\,dy=\int_1^2 x^2\Big|_{x=\sqrt{y-1}}^{x=\sqrt{\frac{y}{2}}}+yx\Big|_{x=\sqrt{y-1}}^{x=\sqrt{\frac{y}{2}}}dy$$

$4\sqrt{2}$

$-\frac{y}{2}$

$$=\int_1^2\frac{y}{2}-y+1+y\sqrt{\frac{y}{2}}-y\sqrt{y-1}\,dy=\frac{-y^2}{4}\Big|_{y=1}^{y=2}+y\Big|_{y=1}^{y=2}+\frac{8}{\sqrt{2}}\frac{y^{\frac{5}{2}}}{5}\Big|_{y=1}^{y=2}+\frac{2}{5}(y-1)^{\frac{5}{2}}\Big|_{y=1}^{y=2}+\frac{2}{3}(y-1)^{\frac{3}{2}}\Big|_{y=1}^{y=2}$$

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$$=-\frac{1}{4}(4-1)+(2-1)+\frac{8}{5}-\frac{\sqrt{2}}{5}+\frac{2}{5}+\frac{2}{3}\qquad -\frac{\sqrt{2}}{5}+\frac{47}{60}$$

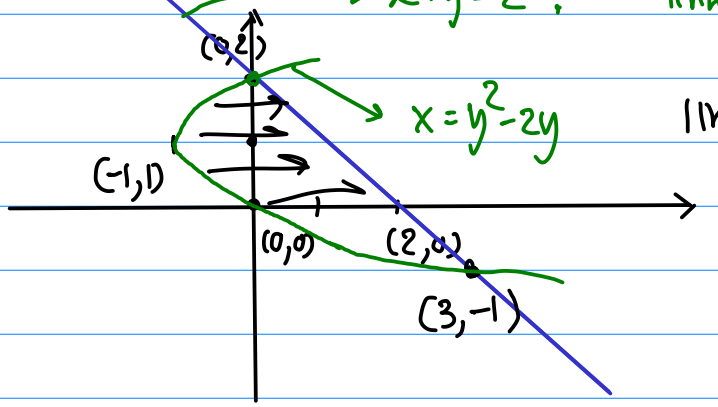
$$\iint_{D_1}(2x+y)dA=\frac{2\sqrt{2}}{5}-\frac{\sqrt{2}}{5}+\frac{17}{60}-\frac{\sqrt{2}}{5}+\frac{47}{60}=\frac{64}{60}=\frac{16}{15}$$

$x = (y-1)^2 - 1$

$y^2 - 2y + 1 - 1$

ตัวอย่าง จงหาพ.ท.ของ D ซึ่งปิดล้อมด้วย $x+y=2$, $x=y^2-2y$

วิธีทำ



$x+y=2$: แทน $x=0 \Rightarrow y=2$, แทน $y=0 \Rightarrow x=2$

$x=y^2-2y$ แทน $x=0 \Rightarrow y=0$, แทน $y=1 \Rightarrow x=-1$

หาจุดตัด

$x+y=2 \therefore x=2-y$

$2-y = y^2-2y$

$0 = y^2-y-2 = (y-2)(y+1)$

ดังนั้น $y=2$ และ $y=-1$

$$D=\{(x,y) \mid y^2-2y \le x \le 2-y,\ -1 \le y \le 2\}$$

$$\iint_D 1\,dA = \int_{-1}^{2}\int_{y^2-2y}^{2-y} 1\,dx\,dy = \int_{-1}^{2} x\Big|_{x=y^2-2y}^{x=2-y} dy = \int_{-1}^{2} 2-y-y^2+2y\,dy$$

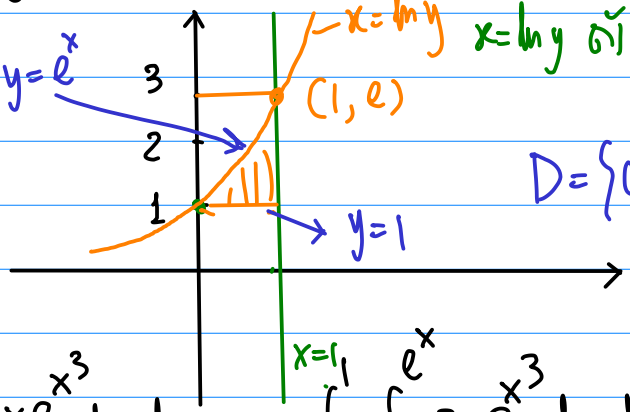
($2y-y=y$)

$$= 2y\Big|_{y=-1}^{y=2} + \frac{y^2}{2}\Big|_{y=-1}^{y=2} - \frac{y^3}{3}\Big|_{y=-1}^{y=2} = 2(2-(-1)) + \frac{1}{2}(4-(-1)) - \frac{1}{3}(8-(-1))$$

$$= 6 + \frac{3}{2} - 3 = 3 + \frac{3}{2} = \frac{9}{2}$$

ตัวอย่าง $\int_1^e \int_{\ln y}^{1} \frac{3xe^{x^3}}{y}\,dx\,dy \Rightarrow D=\{(x,y) \mid \ln y \le x \le 1,\ 1 \le y \le e\}$

($x=\ln y$, $x=1$, $e \approx 2.718$)



$x=\ln y$ ถ้า $y=1$ แล้ว $x=0$

$y=e$ แล้ว $x=1$

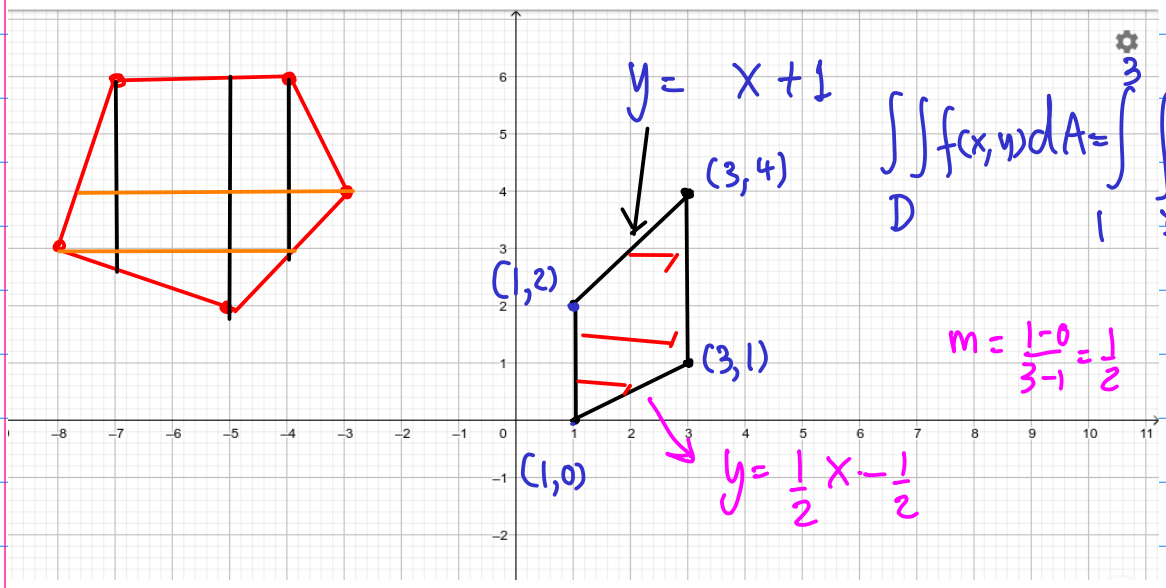
$$D=\{(x,y) \mid 1 \le y \le e^x,\ 0 \le x \le 1\}$$

$$\int_1^e \int_{\ln y}^{1} \frac{3xe^{x^3}}{y}\,dx\,dy = \int_0^1 \int_1^{e^x} \frac{3xe^{x^3}}{y}\,dy\,dx$$

($x=1$)

$\iint_D f(x,y)\,dA$ จุดสร้าง $\square$ D คือ $(1,0), (3,1), (3,4), (1,2)$

$m = \frac{4-2}{3-1} = 1$ $D = \{(x,y) \mid \frac{x}{2} - \frac{1}{2} \le y \le x+1, 1 \le x \le 3\}$



$$\iint_D f(x,y)\,dA = \int_1^3 \int_{\frac{x}{2}-\frac{1}{2}}^{x+1} f(x,y)\,dy\,dx$$

$m = \frac{1-0}{3-1} = \frac{1}{2}$