High order ODE

$$y^{(n)} + P_1(x)y^{(n-1)} + \ldots + P_{n-1}(x)y' + P_n(x)y = R(x) \qquad : Nonhomogeneous$$

$$y^{(n)} + P_1(x)y^{(n-1)} + \ldots + P_{n-1}(x)y' + P_n(x)y = 0 \qquad : Homogeneous$$

$$Complete \quad soln.(y) = Complementary \quad soln.(y_c) + Particular \quad soln.(y_p)$$

$$y = [c_1 y_1 + c_2 y_2 + \ldots + c_n y_n] + y_p$$

$y_1, y_2, \ldots, y_n$: *independent particular solution*

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots\cdots\cdots & y_n \\ y_1' & y_2' & \cdots\cdots\cdots & y_n' \\ . & . & \cdots\cdots\cdots & . \\ y_1^{(n)} & y_2^{(n)} & \cdots\cdots\cdots & y_n^{(n)} \end{vmatrix} \neq 0$$

Constant coefficient High order ODE

$$y^{(n)} + a_1 y^{(n-1)} + ... + a_{n-1} y' + a_n y = R(x) \quad : Nonhomogeneous$$

$$y^{(n)} + a_1 y^{(n-1)} + ... + a_{n-1} y' + a_n y = 0 \quad : Homogeneous$$

If the solution $y = Ae^{mx}$ *then characteristic eq. will be:*

$$m^n + a_1 m^{(n-1)} + ... + a_{n-1} m + a_n = 0$$

Complementary solution (y_c):

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + ... + c_n e^{m_n x} \quad : \quad \forall m_k \neq m_l$$

If $m_1 = m_2 = ... = m_i \Rightarrow$ *multiplicity* $= i$

Finding Complete Solution for Constant Homogeneous n^{th} ODE

$$y^{(n)}+a_1y^{(n-1)}+\ldots+a_{n-1}y'+a_ny = 0$$

Step 1: Form characteristic eq.

$$m^n+a_1m^{n-1}+\ldots+a_{n-1}m'+a_n = 0$$

Step 2: Solve $m_1, m_2,\ldots.m_n$

Check 1: multiplicity > 1 ?

Check 2: m is complex ?

Step 3: $y_c=c_1y_1+c_2y_2+\ldots+c_ny_n$

Constant coefficient High order ODE

$$y^{(n)}+a_1 y^{(n-1)}+\ldots+a_{n-1}y'+a_n y = 0$$

Case 1: Simple root (multiplicity = 1)

or $\forall m_k \neq m_l \Rightarrow y_k = exp(m_k x)$

Case 2: Multiplicity = i or $m_1=m_2=....=m_i$

$y_1=exp(m_1 x)$, $y_2=(x)exp(m_2 x)$, $\ldots.y_i=(x)^{i-1}exp(m_i x)$

Case 3: m is complex number:

$m_1,m_2=a\pm jb \Rightarrow y_1=e^{ax}cos(bx)$, $y_2=e^{ax}sin(bx)$

Case 4: m is complex number and multiplicity = i

$m_1,m_2=a\pm jb$

$y_1=e^{ax}cos(bx)$, $y_{i+1}=e^{ax}sin(bx)$

$y_2=(x)e^{ax}cos(bx)$, $y_{i+2}=(x)e^{ax}sin(bx)$

… …

$y_i=(x)^{i-1}e^{ax}cos(bx)$, $y_{2i}=(x)^{i-1}eaxsin(bx)$

Examples: Constant coefficient High order ODE

$Ex(4-1):\qquad y''' - 3y'' - 10y' + 24y = 0$

$\Rightarrow\qquad m^3 - 3m^2 - 10m + 24 = (m-2)(m+3)(m-4) = 0$

$\Rightarrow\qquad y = c_1 e^{2x} + c_1 e^{-3x} + c_1 e^{4x}$ ...###

$Ex(4-2):\qquad y^{(4)} - 4y''' + 6y'' - 4y' + y = 0$

$\Rightarrow\qquad m^4 - 4m^3 + 6m^2 - 4m + 1 = (m-1)^4 = 0$

$\Rightarrow\qquad y_1 = e^x,\quad y_2 = xe^x,\quad y_3 = x^2 e^x,\quad y_4 = x^3 e^x$

$\therefore\qquad y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + c_4 x^3 e^x$ ...###

$Ex(4-3):\qquad y^{(5)} - 2y^{(4)} + 8y'' - 12y' + 8y = 0$

$\Rightarrow\qquad m^5 - 2m^4 + 8m^2 - 12m + 8 = (m+2)(m^2 - 2m + 2)^2 = 0$

$\Rightarrow\qquad m_1 = -2,\qquad m_2 = m_3 = 1 + j,\qquad m_4 = m_5 = 1 - j$

$\Rightarrow y_1 = e^{-2x},\quad y_2 = e^x\cos(x),\quad y_3 = e^x\sin(x),\quad y_4 = xe^x\cos(x),\quad y_5 = xe^x\sin(x)$

$\therefore\qquad y = c_1 e^{-2x} + c_2 e^x\cos(x) + c_3 e^x\sin(x) + c_4 x e^x\cos(x) + c_5 x e^x\sin(x)$...###

Finding Complete Solution for Constant Non-homogeneous n[th] ODE

$$y^{(n)}+a_1 y^{(n-1)}+\ldots+a_{n-1}y'+a_n y = f(x)$$

$$y = y_c + y_p$$

y_c : *Complementary function* y_p : *Particular integral*

Step 1: Find complementary function from homogeneous eq.

$$y_c=c_1y_1+c_2y_2+\ldots+c_ny_n$$

Step 2: Find particular solution

2.1 undetermined coefficients

2.2 variation of parameters

Examples: const. coef. Non-homo. high order ODE

$$Ex(4-4): \qquad y''' + 5y'' + 9y' + 5y = 3e^{2x}$$

$$Step.1 \Rightarrow y_c: \qquad m^3 + 5m^2 + 9m + 5 = (m+1)(m^2 + 4m + 5) = 0$$

$$\Rightarrow \quad m = -1, -2 \pm j$$

$$\therefore \qquad y_c = c_1 e^{-x} + e^{-2x}[c_2 \cos(x) + c_3 \sin(x)]$$

$$Step.2 \Rightarrow y_p: \qquad f(x) = 3e^{2x} \qquad \Rightarrow \qquad y_p = Ae^{2x}$$

$$y_p' = 2Ae^{2x} \qquad , \qquad y_p'' = 4Ae^{2x}, \qquad y_p''' = 8Ae^{2x}$$

$$\therefore \quad 8Ae^{2x} + 5(4Ae^{2x}) + 9(2Ae^{2x}) + 5(Ae^{2x}) = 3e^{2x} \quad \therefore \quad A = 1/17$$

$$y = c_1 e^{-x} + e^{-2x}[c_2 \cos(x) + c_3 \sin(x)] + (1/17)e^{2x}$$

Examples: Const. coef. Non-homo. high order ODE

$$Ex(4-5): \qquad (D^4+8D^2+16)y=\sin(x)$$

$$Step.1 \Rightarrow y_c: \quad m^4+8m^2+16=(m^2+4)^2=0$$

$$\Rightarrow \quad m_1=m_2=2j \qquad , \qquad m_3=m_4=-2j$$

$$\therefore \quad y_c=c_1\cos(2x)+c_2\sin(2x)+c_3x\cos(2x)+c_4x\sin(2x)$$

$$Step.2 \Rightarrow y_p: \quad f(x)=\sin(x) \qquad \Rightarrow \qquad y_p=A\cos(x)+B\sin(x)$$

$$y_p''=-A\cos(x)-B\sin(x), \qquad y_p''''=A\cos(x)+B\sin(x)$$

$$\therefore \quad (A\cos(x)+B\sin(x))+8(-A\cos(x)-B\sin(x))$$

$$+16(A\cos(x)+B\sin(x))=\sin(x)$$

$$\therefore \qquad 9A\cos(x)+9B\sin(x)=\sin(x) \qquad \Rightarrow A=0, B=1/9$$

$$\therefore \qquad y_p=(1/9)\sin(x)$$

$$y=c_1\cos(2x)+c_2\sin(2x)+c_3x\cos(2x)+c_4x\sin(2x)+(1/9)\sin(x) ...\#\#$$

Reduction of order

Method 1 :Using operator D

$$(D-m_1)(D-m_2)\ldots(D-m_n)y = f(x)$$

$$(D-m_2)\ldots(D-m_n)y = u$$

$$\therefore \quad (D-m_1)u = f(x)$$

Method 2 : Using lower ODE

$$u_1 = y,$$

$$u_2 = du_1 = y'$$

$$u_3 = du_2 = y''$$

.

.

Examples of Reduction of order

$Ex(4-6): \qquad y''' + 5y'' + 9y' + 5y = 3e^{2x}$

$\Rightarrow \qquad (D^3 + 5D^2 + 9D + 5)y = (D+1)(D^2 + 4D + 5)y = 3e^{2x}$

$assume: \qquad (D^2 + 4D + 5)y = (D + 2 - j)(D + 2 + j)y = u$

$\therefore \qquad (D+1)u = \frac{du}{dx} + u = 3e^{2x} \text{..........} integrating \quad factor = e^{\int 1dx} = e^x$

$\Rightarrow \quad e^x \frac{du}{dx} + e^x u = e^x 3e^{2x} \quad \Rightarrow \quad \frac{d}{dx}(ue^x) = 3e^{3x}$

$\Rightarrow \int \Rightarrow ue^x = e^{3x} + C \qquad \Rightarrow \qquad u = e^{2x} + Ce^{-x}$

$From \quad (D + 2 - j)(D + 2 + j)y = u \therefore \qquad (D + 2 - j)(D + 2 + j)y = e^{2x} + Ce^{-x}$

$assume: \quad y_p = c_1 e^{2x} + c_2 Ce^{-x}, \quad y_p' = 2c_1 e^{2x} - c_2 Ce^{-x}, \quad y_p'' = 4c_1 e^{2x} + c_2 Ce^{-x}$

$\therefore \qquad (D^2 + 4D + 5)y = u$

$(4c_1 e^{2x} + c_2 Ce^{-x}) + 4(2c_1 e^{2x} - c_2 Ce^{-x}) + 5(c_1 e^{2x} + c_2 Ce^{-x}) = e^{2x} + Ce^{-x}$

$\Rightarrow \qquad 17c_1 e^{2x} + 2c_2 Ce^{-x} = e^{2x} + Ce^{-x}, \qquad c_1 = 1/17, c_2 = 1/2$

$\therefore \qquad y = y_c + y_p = [e^{-2x}(A\cos(x) + B\sin(x)] + (\frac{1}{17}e^{2x} + \frac{1}{2}Ce^{-x})...\#\#\#$

Examples of Reduction of order

$Ex(4-7): \; Find \;\; complete \;\; solution \;\; of \;\; (D^4 + 8D^2 + 16)y = \sin(x)$

$$\Rightarrow \quad (D^2+4)^2 y = (D^2+4)(D^2+4)y = \sin(x)$$

$$assume: \; (D^2+4)y = u \quad \Rightarrow \quad (D^2+4)u = \sin(x)$$

$$\therefore \; u_c = A\cos(2x) + B\sin(2x), \;\; u_p = c_1\cos(x) + c_2\sin(x)$$

$$u_p' = -c_1\sin(x) + c_2\cos(x), \;\; u_p'' = -c_1\cos(x) - c_2\sin(x)$$

$$\therefore \; [-c_1\cos(x) - c_2\sin(x)] + 4[c_1\cos(x) + c_2\sin(x)] = \sin(x)$$

$$\therefore \; 3c_1 = 0 \Rightarrow c_1 = 0, \quad 3c_2 = 1 \Rightarrow c_2 = \frac{1}{3}, \quad u_p = \frac{1}{3}\sin(x)$$

$$\therefore \; u = [A\cos(2x) + B\sin(2x)] + (1/3)\sin(x)$$

$$from \;\; (D^2+4)y = u \quad \Rightarrow (D^2+4)y = [A\cos(2x) + B\sin(2x)] + (1/3)\sin(x)$$

$$y_c = c_{c1}\cos(2x) + c_{c2}\sin(2x), \quad y_p = c_{p1}x\cos(2x) + c_{p2}x\sin(2x) + c_{p3}\sin(x)$$

$$y_p' = c_{p1}(-2x\sin(2x) + \cos(2x)) + c_{p2}(2x\cos(2x) + \sin(2x)) + c_{p3}\cos(x)$$

$$y_p'' = c_{p1}[-4x\cos(2x) - 2\sin(2x) - 2\sin(2x)] + c_{p2}[-4x\sin(2x) + 2\cos(2x) + \cos(2x)] - c_{p3}\sin(x)$$

$$= (-4xc_{p1} + 3c_{p2})\cos(2x) + (-4c_{p1} - 4xc_{p2})\sin(2x) - c_{p3}\sin(x)$$

$$\therefore \; (D^2+4)y = 3c_{p2}\cos(2x) - 4c_{p1}\sin(2x) + 3c_{p3}\sin(x) = [A\cos(2x) + B\sin(2x)] + (1/3)\sin(x)$$

$$\Rightarrow c_{p1} = -(B/4), c_{p2} = (A/3), c_{p3} = (1/9) \quad \Rightarrow \quad y_p = -\frac{B}{4}x\cos(2x) + \frac{A}{3}x\sin(2x) + \frac{1}{9}\sin(x)$$

$$\therefore y = [c_{c1}\cos(2x) + c_{c2}\sin(2x)] + [-\frac{B}{4}x\cos(2x) + \frac{A}{3}x\sin(2x) + \frac{1}{9}\sin(x)] ...\#\#\#$$

Examples of Reduction of order

$$Ex(4-8): \quad Reduce \ the \ order \ of \quad y''' + 5y'' + 9y' + 5y = 3e^{2x}$$

$$assume: \quad u_1 = y, \quad u_2 = u_1' = y', \quad u_3 = u_2' = y'' \quad \therefore \quad u_3' = y'''$$

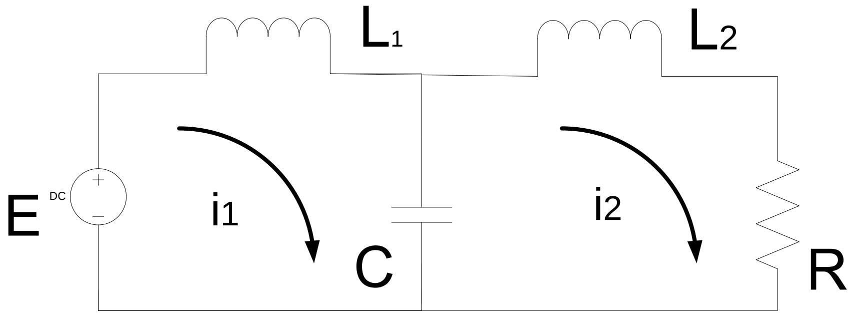
$$\Rightarrow \quad u_3' = y''' = -5y'' - 9y' - 5y + 3e^{2x} = -5u_3 - 9u_2 - 5u_1 + 3e^{2x}$$

$$\Rightarrow \frac{du_1}{dx} = (0)u_1 \quad + (1)u_2 \quad + (0)u_3 \quad + 0$$

$$\Rightarrow \frac{du_2}{dx} = (0)u_1 \quad + (0)u_2 \quad + (1)u_3 \quad + 0$$

$$\Rightarrow \frac{du_3}{dx} = (-5)u_1 + (-9)u_2 + (-5)u_3 + 3e^{2x}$$

$$\left(\frac{d}{dx}\right)\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -9 & -5 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3e^{2x} \end{bmatrix} \ldots\#\#\#$$



การประยุกต์ทางวิศวกรรมไฟฟ้า

$$Find \quad i_2 \quad , \quad E = 10V, \quad L_1 = 8H, \quad C = 0.123F, \quad L_2 = 1H, \quad R = 5ohm,$$

$$t = 0, \quad i_1 = i_2 = 0, V_c = 0$$

$$\Rightarrow 1: \qquad V_{L1} + V_{L2} + V_R = L_1 i_1' + L_2 i_2' + R i_2 = E$$

$$\Rightarrow 2: \qquad V_C = \frac{1}{C}\int (i_1 - i_2) dt + V_{C0} = L_2 i_2' + R i_2$$

$$\frac{d}{dt} : \Rightarrow \quad (i_1 - i_2) = L_2 C i_2'' + RC i_2' \Rightarrow \qquad i_1 = i_2 + L_2 C i_2'' + RC i_2'$$

$$i_1' = i'_2 + L_2 C i_2''' + RC i_2''$$

$$\therefore \quad L_1 i_1' + L_2 i_2' + R i_2 = L_1 (i'_2 + L_2 C i_2''' + RC i_2'') + L_2 i_2' + R i_2 = E$$

$$\Rightarrow \qquad L_1 L_2 C i_2''' + L_1 RC i_2'' + (L_1 + L_2) i_2' + R i_2 = E$$

$$\Rightarrow \qquad i_2''' + 5 i_2'' + 9 i_2' + 5 i_2 = 10$$

$$\Rightarrow \qquad (D+1)(D^2 + 4D + 5) i_2 = 10$$

......