Boundary-Value Problem

Scope: Linear & constant coefficient PDE2
with boundary condition

Solving method”
1. Method of separation of variables
2. Transform method

Method of separation of variables

1. สมมุติตัวแปรตามเป็นผลคูณของตัวแปรต้น :

$$z(x,y) = X(x)Y(y)$$

2. หาอนุพันธ์ย่อยของตัวแปรตาม :

$$\frac{\partial z}{\partial x} = X'Y, \qquad \frac{\partial^2 z}{\partial x^2} = X''Y,$$

$$\frac{\partial z}{\partial y} = XY', \qquad \frac{\partial^2 z}{\partial y^2} = XY'',$$

3. แยกตัวแปรต้นออกจากกัน

$$F(X, X', X'') = G(Y, Y', Y'') = const.$$

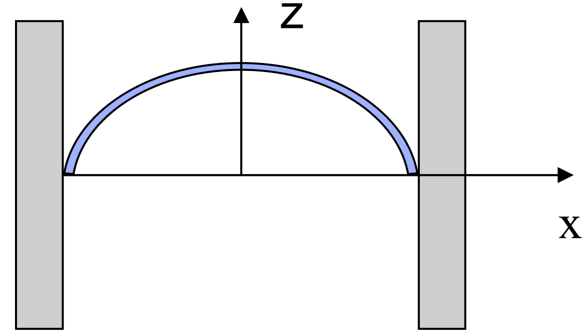
4. หาคำตอบแยก

$$X = \ldots, \qquad Y = \ldots, \qquad \Rightarrow \qquad z = X(x)Y(y)$$

Ex(15-1) การแกว่งของเชือกเป็นตามสมการคลื่น

จงหาการกระจัด z ในเทอมของ x, t เมื่อ

1. $z(0,t)=z(L,t)=0$
2. $z(x,0)=f(x) \ldots 0\le x \le L$
3. $[\partial z/\partial t] = g(x) \ldots\ 0\le x \le L\ ,\ t=0,$



$$wave\ \ equation: \qquad \frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2},$$

$$Step1. \qquad assume: \qquad z(x,t) = X(x)T(t)$$

$$Step2. \qquad \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 X}{\partial x^2}T = \frac{d^2 X}{dx^2}T = X''T, \qquad \frac{\partial^2 z}{\partial t^2} = X\frac{\partial^2 T}{\partial t^2} = X\frac{d^2 T}{dt^2} = XT''$$

$$\therefore \qquad X''T = (\frac{1}{c^2})XT'' \qquad \Rightarrow \qquad c^2\frac{X''}{X} = \frac{T''}{T} = const. = \begin{Bmatrix} +\omega^2 \\ 0 \\ -\omega^2 \end{Bmatrix}, \qquad [\ or \qquad \frac{c^2}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \begin{Bmatrix} +\omega^2 \\ 0 \\ -\omega^2 \end{Bmatrix}\ ]$$

$$case\ \ 1: \qquad const. = +\omega^2$$

$$1) \qquad c^2 X''/X = \omega^2, \qquad \Rightarrow \qquad X'' - (\omega/c)^2 X = 0, \qquad X = A_1 e^{\omega x/c} + B_1 e^{-\omega x/c} \ = a_1\cosh(\omega x/c) + b_1\sinh(\omega x/c)$$

$$2) \qquad T''/T = \omega^2, \qquad \Rightarrow \qquad T'' - \omega^2 T = 0, \qquad T = C_1 e^{\omega t} + D_1 e^{-\omega t} \ = \ c_1\cosh(\omega t) + d_1\sinh(\omega t)$$

$$at \quad x = 0, \quad z(0,t) = (A_1 + B_1)T = 0$$

$$at \quad x = L, \quad z(L,t) = (A_1 e^{\omega L/c} + B_1 e^{-\omega L/c})T = 0, \qquad \Rightarrow \qquad A_1 = B_1 = 0, \qquad \Rightarrow \qquad const. \ne +\omega^2$$

$$\therefore \qquad X''T = \left(\frac{1}{c^2}\right)XT'' \qquad \Rightarrow \qquad c^2\frac{X''}{X} = \frac{T''}{T} = const. = \begin{Bmatrix} +\omega^2 \\ 0 \\ -\omega^2 \end{Bmatrix},$$

$case \;\; 2: \qquad const. = 0$

1) $\quad c^2 X''/X = 0, \quad \Rightarrow \quad X'' = 0, \quad X = A_2 x + B_2$

2) $\quad T''/T = 0, \quad \Rightarrow \quad T'' = 0, \quad T = C_2 t + D_2$

$at \quad x = 0, \quad z(0,t) = (0 + B_2)T = 0, \quad \Rightarrow \quad B_2 = 0$

$at \quad x = L, \quad z(L,t) = (A_2 L + (0))T = 0, \quad \Rightarrow \quad A_2 = 0,$

$\therefore \qquad const. \neq 0$

$case\ \ 3: \qquad const. = -\omega^2$

$$1)\quad c^2 X''/X = -\omega^2, \quad \Rightarrow \quad X'' + (\omega/c)^2 X = 0, \quad X = A_3\cos(\omega x/c) + B_3\sin(\omega x/c) = a_3 e^{j\omega x/c} + b_3 e^{-j\omega x/c}$$

$$2)\quad T''/T = -\omega^2, \quad \Rightarrow \quad T'' + \omega^2 T = 0, \quad T = C_3\cos(\omega t) + D_3\sin(\omega t) \quad = c_3 e^{j\omega t} + d_3 e^{-j\omega t}$$

$$x = 0, \quad z(0,t) = [A_3 + 0]T = 0, \quad \Rightarrow \quad A_3 = 0$$

$$x = L, \quad z(L,t) = [0 + B_3\sin(\omega L/c)]T = 0, \quad \Rightarrow \quad (\omega L/c) = r\pi, \quad \therefore \quad \omega = r\pi c/L \text{..........} r = 1,2,3,\text{....}$$

$$\therefore \quad z(x,t) = B_3\sin(r\pi x/L)[C_3\cos(r\pi ct/L) + D_3\sin(r\pi ct/L)] = \sin(r\pi x/L)[h_3\cos(r\pi ct/L) + k_3\sin(r\pi ct/L)]$$

$$\therefore \quad z(x,t) = \sum_{r=1}^{\infty} z_r(x,t) \quad = \sum_{r=1}^{\infty}\sin(r\pi x/L)[h_r\cos(r\pi ct/L) + k_r\sin(r\pi ct/L)]$$

$$t = 0, \quad z(x,0) = f(x) = \sum_{r=1}^{\infty}\sin(r\pi x/L)[h_r\cos(0) + k_r\sin(0)] = \sum_{r=1}^{\infty}(h_r)\sin(r\pi x/L)$$

$$\int_0^L f(x)\sin(n\pi x/L)dx = \sum_{r=1}^{\infty}[\int_0^L (h_r)\sin(r\pi x/L)\sin(n\pi x/L)dx] = h_n L/2$$

$$\Rightarrow \quad h_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)dx$$

$$\left[\quad \int_0^L \sin(r\pi x/L)\sin(n\pi x/L)dx = \begin{cases} 0 \text{................} r \neq n \\ L/2 \text{............} r = n \end{cases} \quad \right]$$

$$\therefore \quad \frac{\partial z}{\partial t} = \sum_{r=1}^{\infty}(r\pi c/L)\sin(r\pi x/L)[-h_r\sin(r\pi ct/L) + k_r\cos(r\pi ct/L)]$$

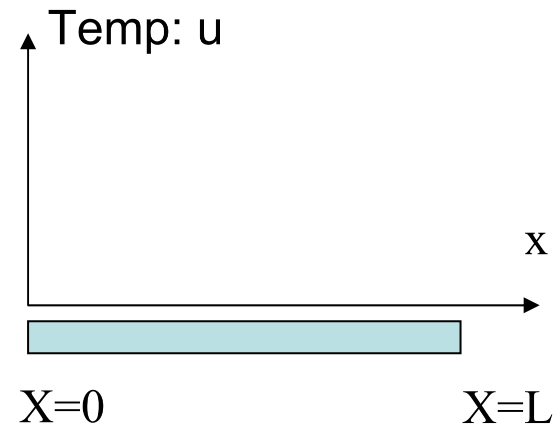
$$t = 0, \quad \left[\frac{\partial z}{\partial t}\right]_{t=0} = g(x) \quad = \sum_{r=1}^{\infty}(r\pi c/L)\sin(r\pi x/L)[0 + k_r] = (\pi c/L)\sum_{r=1}^{\infty}(rk_r)\sin(r\pi x/L)$$

$$\int_0^L g(x)\sin(n\pi x/L)dx = (\pi c/L)\sum_{r=1}^{\infty}[\int_0^L (rk_r)\sin(r\pi x/L)\sin(n\pi x/L)dx] = (\pi c/L)(nk_n L/2) = (\pi c/2)(nk_n)$$

$$\Rightarrow \quad k_n = \frac{2}{n\pi c}\int_0^L g(x)\sin(n\pi x/L)dx$$

Ex(15-2) การนำความร้อนเป็นตามสมการนำความร้อน จงหาอุณหภูมิ u ที่ตำแหน่งต่างๆ ในเทอมของ x, t เมื่อ

1. u(0,t)=z(L,t)=0…..t≥0
2. u(x,0)=f(x)………0≤x ≤L



$$Equation: \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t},$$

$$Step1. \quad assume: \quad u(x,t) = X(x)T(t)$$

$$Step2. \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 X}{\partial x^2}T = \frac{d^2 X}{dx^2}T = X''T, \qquad \frac{\partial u}{\partial t} = X\frac{\partial T}{\partial t} = XT'$$

$$\therefore \quad X''T = \frac{1}{k}XT' \quad \Rightarrow \quad \frac{X''}{X} = \frac{1}{k}\frac{T'}{T} = const. = \begin{Bmatrix} +\omega^2 \\ 0 \\ -\omega^2 \end{Bmatrix}, \quad \left[or \quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{kT}\frac{\partial T}{\partial t} = \begin{Bmatrix} +\omega^2 \\ 0 \\ -\omega^2 \end{Bmatrix} \right]$$

$$case \quad 1: \quad const. = +\omega^2$$

$$1) \quad X''/X = \omega^2, \quad \Rightarrow \quad X'' - \omega^2 X = 0, \quad X = A_1 e^{\omega x} + B_1 e^{-\omega x} \quad = a_1\cosh(\omega x) + b_1\sinh(\omega x)$$

$$2) \quad T'/T = k\omega^2, \quad \Rightarrow \quad T' - k\omega^2 T = 0, \quad T = C_1 e^{k\omega^2 t}$$

$$at \quad x = 0, \quad z(0,t) = (A_1 + B_1)T = 0$$

$$at \quad x = L, \quad z(L,t) = (A_1 e^{\omega L/c} + B_1 e^{-\omega L/c})T = 0, \quad \Rightarrow \quad A_1 = B_1 = 0, \quad \Rightarrow \quad const. \neq +\omega^2$$

$$case \quad 2: \quad const. = 0$$

$$1) \quad X''/X = 0, \quad \Rightarrow \quad X'' = 0, \quad X = A_1 x + B_1$$

$$2) \quad T'/T = 0, \quad \Rightarrow \quad T' = 0, \quad T = C_1$$

$$at \quad x = 0, \quad z(0,t) = (0 + B_1)T = 0 \quad \Rightarrow \quad B_1 = 0$$

$$at \quad x = L, \quad z(L,t) = (A_1 L + B_1)T = 0, \quad \Rightarrow \quad A_1 = 0, \quad \Rightarrow \quad const. \neq 0$$

$$case \quad 3: \qquad const. = -\omega^2$$

$$1) \qquad X''/X = -\omega^2, \qquad \Rightarrow \qquad X''+\omega^2 X = 0, \qquad X = A_3\cos(\omega x) + B_3\sin(\omega x) = a_3 e^{j\omega x/c} + b_3 e^{-j\omega x/c}$$

$$2) \qquad T'/kT = -\omega^2, \qquad \Rightarrow \qquad T'+k\omega^2 T = 0, \qquad T = C_3 e^{-k\omega^2 t}$$

$$x = 0, \quad u(0,t) = [A_3 + 0]T = 0, \qquad \Rightarrow \qquad A_3 = 0$$

$$x = L, \quad u(L,t) = [0 + B_3\sin(\omega L)]T = 0, \qquad \Rightarrow \qquad (\omega L) = r\pi, \qquad \therefore \qquad \omega = r\pi/L..........r = 1,2,3,....$$

$$\therefore \qquad u(x,t) = B_3\sin(\omega x)C_3 e^{-k\omega^2 t} = B_3\sin(r\pi x/L)C_3 e^{-kr^2\pi^2 t/L^2} = D\sin(r\pi x/L)e^{-kr^2\pi^2 t/L^2}...........D = B_3C_3$$

$$\therefore \qquad u(x,t) = \sum_{r=1}^{\infty} u_r(x,t) \qquad = \sum_{r=1}^{\infty} D_r\sin(r\pi x/L)e^{-kr^2\pi^2 t/L^2}$$

$$t = 0, \quad u(x,0) = f(x) = \sum_{r=1}^{\infty} D_r\sin(r\pi x/L)$$

$$\int_0^L f(x)\sin(n\pi x/L)dx = \sum_{r=1}^{\infty}[\int_0^L D_r\sin(r\pi x/L)\sin(n\pi x/L)dx] = D_n L/2$$

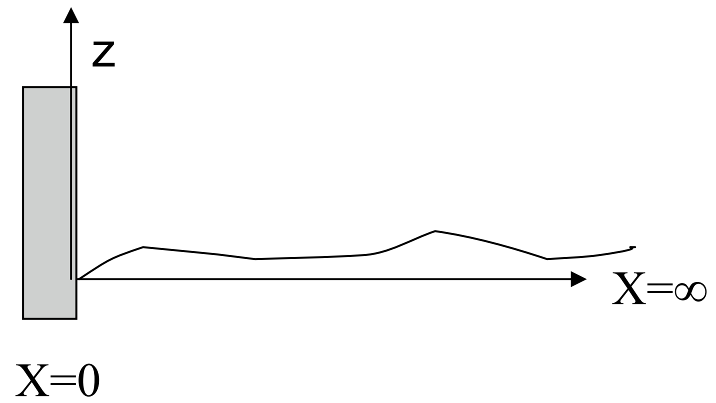
$$\Rightarrow \qquad D_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)dx$$

$$[\qquad \int_0^L \sin(r\pi x/L)\sin(n\pi x/L)dx = \begin{cases} 0.................r \neq n \\ L/2............r = n \end{cases} \qquad]$$

Ex(15-3) การแกว่งของเชือกเป็นตามสมการคลื่น

จงหาการกระจัด z ในเทอมของ x, t เมื่อ

1. $z(0,t)=0$
2. $z(x,t)<\infty$……….$x>0,\ t>0$
3. $z(x,0)=f(x)$……$0\le x\le\infty,\ t=0$
4. $[\partial z/\partial t]=g(x)$…..$0\le x\le\infty,\ t=0,$



$$wave\ \ equation: \quad \frac{\partial^2 z}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2},$$

$$Step1. \quad assume: \quad z(x,t)=X(x)T(t)$$

$$Step2. \quad \frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 X}{\partial x^2}T=\frac{d^2X}{dx^2}T=X''T, \qquad \frac{\partial^2 z}{\partial t^2}=X\frac{\partial^2 T}{\partial t^2}=X\frac{d^2T}{dt^2}=XT''$$

$$\therefore \quad X''T=(\frac{1}{c^2})XT'' \quad \Rightarrow \quad c^2\frac{X''}{X}=\frac{T''}{T}=const.=\begin{Bmatrix}+\omega^2\\0\\-\omega^2\end{Bmatrix}, \quad [\ or \quad \frac{c^2}{X}\frac{\partial^2 X}{\partial x^2}=\frac{1}{T}\frac{\partial^2 T}{\partial t^2}=\begin{Bmatrix}+\omega^2\\0\\-\omega^2\end{Bmatrix}\ ]$$

$$case\ \ 1: \quad const.=+\omega^2$$

1) $c^2X''/X=\omega^2, \quad \Rightarrow \quad X''-(\omega/c)^2X=0, \quad X=A_1e^{\omega x/c}+B_1e^{-\omega x/c}=a_1\cosh(\omega x/c)+b_1\sinh(\omega x/c)$

2) $T''/T=\omega^2, \quad \Rightarrow \quad T''-\omega^2T=0, \quad T=C_1e^{\omega t}+D_1e^{-\omega t}=c_1\cosh(\omega t)+d_1\sinh(\omega t)$

$$at \quad x=0, \quad z(0,t)=(A_1+B_1)T=0 \quad \Rightarrow \quad A_1=-B_1$$

$$z(x,t)=A_1(e^{\omega x/c}-e^{-\omega x/c})T$$

$$x=\infty: \quad z(\infty,t)=\infty \quad but \quad z(x,t)<\infty$$

$$\therefore \quad const.\neq+\omega^2$$

$$\therefore \qquad X''T = \left(\frac{1}{c^2}\right)XT'' \qquad \Rightarrow \qquad c^2\frac{X''}{X} = \frac{T''}{T} = const. = \begin{Bmatrix} +\omega^2 \\ 0 \\ -\omega^2 \end{Bmatrix},$$

$$case \quad 2: \qquad const. = 0$$

$$1) \qquad c^2 X''/X = 0, \qquad \Rightarrow \qquad X'' = 0, \qquad X = A_2 x + B_2$$

$$2) \qquad T''/T = 0, \qquad \Rightarrow \qquad T'' = 0, \qquad T = C_2 t + D_2$$

$$at \quad x = 0, \quad z(0,t) = (0 + B_2)T = 0, \qquad \Rightarrow \qquad B_2 = 0$$

$$at \quad x = \infty, \quad z(\infty,t) = (A_2 x + (0))T = \infty, \qquad but \qquad z(x,t) \neq \infty .....\forall x,t$$

$$\therefore \qquad const. \neq 0$$

$$case\quad 3: \qquad const. = -\omega^2$$

$$1)\quad c^2 X''/X = -\omega^2, \quad \Rightarrow \quad X'' + (\omega/c)^2 X = 0, \quad X = A_3\cos(\omega x/c) + B_3\sin(\omega x/c) = a_3 e^{j\omega x/c} + b_3 e^{-j\omega x/c}$$

$$2)\quad T''/T = -\omega^2, \quad \Rightarrow \quad T'' + \omega^2 T = 0, \quad T = C_3\cos(\omega t) + D_3\sin(\omega t) \quad = c_3 e^{j\omega t} + d_3 e^{-j\omega t}$$

$$x = 0, \quad z(0,t) = [A_3 + 0]T = 0, \qquad \Rightarrow \qquad A_3 = 0$$

$$\therefore \quad z(x,t) = B_3\sin(\omega x/c)[C_3\cos(\omega t) + D_3\sin(\omega t)] = \sin(\omega x/c)[h_3\cos(\omega t) + k_3\sin(\omega t)]$$

$$h_3, k_3 : F(\omega)$$

$$\Rightarrow \quad z(x,t) = \int_0^\infty z_\omega(x,t)d\omega \quad = \int_0^\infty \sin(\omega x/c)[h_\omega\cos(\omega t) + k_\omega\sin(\omega t)]d\omega$$

$$t = 0, \quad z(x,0) = f(x) = \int_0^\infty \sin(\omega x/c)[h_\omega + k_\omega(0)]d\omega = \int_0^\infty h_\omega\sin(\omega x/c)d\omega$$

$$[\quad Fourier\quad sine\quad transform:$$

$$g(t) = 4\int_0^\infty G(f)\sin(2\pi ft)df \quad \Leftrightarrow \quad G(f) = F_s[g(t)] = \int_0^\infty g(t)\sin(2\pi ft)dt \quad]$$

$$\therefore \quad h_\omega = h(\omega) = \frac{2}{\pi c}\int_0^\infty f(x_1)\sin(\omega x_1/c)dx_1$$

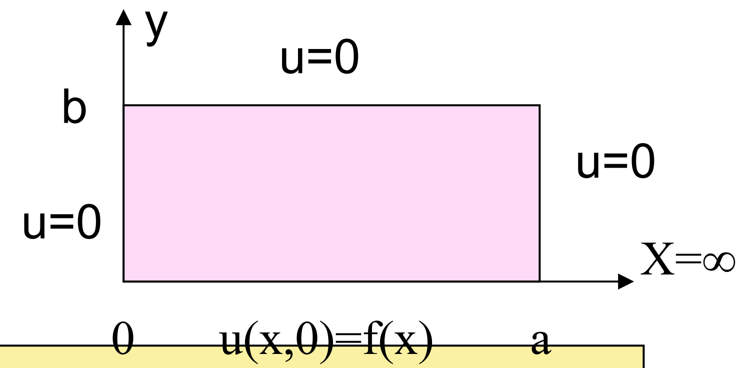
$$\therefore \quad \frac{\partial z}{\partial t} = \int_0^\infty \sin(\omega x/c)\frac{\partial}{\partial t}[h_\omega\cos(\omega t) + k_\omega\sin(\omega t)]d\omega = \int_0^\infty \sin(\omega x/c)[-h_\omega\omega\sin(\omega t) + k_\omega\omega\cos(\omega t)]d\omega$$

$$t = 0, \quad \left[\frac{\partial z}{\partial t}\right]_{t=0} = g(x) \quad = \int_0^\infty \sin(\omega x/c)[0 + k_\omega\omega]d\omega = \int_0^\infty k_\omega\omega\sin(\omega x/c)]d\omega$$

$$\Rightarrow \quad k_\omega = k(\omega) = \frac{2}{\pi\omega c}\int_0^\infty g(x_1)\sin(\omega x_1/c)dx_1$$

Ex(15-6) ศักย์ไฟฟ้าสถิตในบริเวณ 2 มิติที่ไม่มีประจุไฟฟ้าอิสระ จะเป็นไปตามสมการของลาปาส ถ้าให้กล่องตัวนำมีศักย์ไฟฟ้าเป็น

1. u(x,y)=0…….x=0, 0 ≤y ≤b
2. u(x,y)=0…….x=a, 0 ≤y ≤b
3. u(x,y)=0…….y=b, 0 ≤x ≤a
4. u(x,y)=f(x)….y=0, 0 <x <a จงหาศักย์ u(x,y)



$$Laplace\ \ equation: \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad , \quad Step1. \quad assume: \quad u(x,t)=X(x)Y(t)$$

$$Step2. \quad \therefore \quad X''Y+XY''=0 \quad \Rightarrow \quad \frac{X''}{X}=-\frac{Y''}{Y}=-\omega^2$$

$$-\Rightarrow X''/X=-\omega^2,\ \ X=A\cos(\omega x)+B\sin(-\omega x), \qquad -Y''/Y=-\omega^2,\ \ Y=C\cosh(\omega y)+D\sinh(\omega y)$$

$$\therefore \quad u(x,y)=[A\cos(\omega x)+B\sin(\omega x)][C\cosh(\omega y)+D\sinh(\omega y)]$$

1) $x=0,\quad u(0,y)=0=(A+0)[C\cosh(\omega y)+D\sinh(\omega y)] \quad \Rightarrow \quad A=0$

$$\therefore \ u(x,y)=B\sin(\omega x)[C\cosh(\omega y)+D\sinh(\omega y)] \qquad =\sin(\omega x)[E\cosh(\omega y)+F\sinh(\omega y)]$$

2) $x=a,\quad u(a,y)=0=\sin(\omega a)[E\cosh(\omega y)+F\sinh(\omega y)] \quad \Rightarrow \quad \omega a=r\pi \ \ or \ \ \omega=r\pi/a....[r=1,2,3...]$

$$\therefore \ u(x,y)=\sin(r\pi x/a)[E_r\cosh(r\pi y/a)+F_r\sinh(r\pi y/a)]$$

3) $y=b,\quad u(x,b)=0=\sin(r\pi x/a)[E_r\cosh(r\pi b/a)+F_r\sinh(r\pi b/a)] \quad \Rightarrow \quad E_r=-F_r\sinh(r\pi b/a)/\cosh(r\pi b/a)$

$$\therefore \ u(x,y)=F_r\sin(r\pi x/a)[-\sinh(r\pi b/a)\cosh(r\pi y/a)+\cosh(r\pi b/a)\sinh(r\pi y/a)]/\cosh(r\pi b/a)$$

$$=G_r\sin(r\pi x/a)[\sinh\{r\pi(b-y)/a\}]..G_r=-F_r/\cosh(r\pi b/a)$$

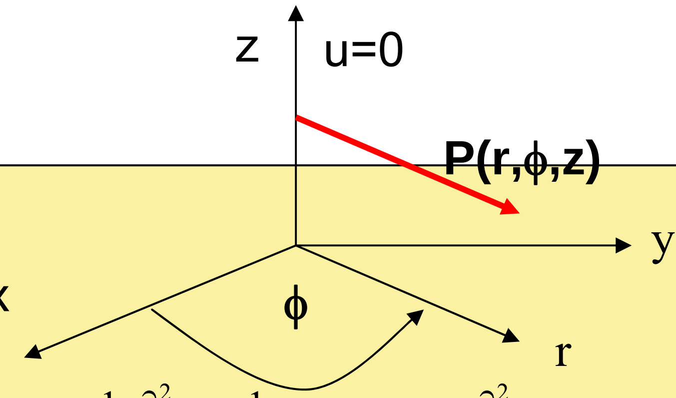
$$\therefore \forall r \quad \Rightarrow u(x,y)=\sum_{r=1}^{\infty}G_r\sin(r\pi x/a)[\sinh\{r\pi(b-y)/a\}]$$

4) $u(x,0)=f(x)=\sum_{r=1}^{\infty}G_r\sin(r\pi x/a)[\sinh\{r\pi(b-y)/a\}],$

$$G_r=\frac{1}{\sinh\{r\pi(b-y)/a\}}\frac{2}{a}\int_0^a f(x)\sin(r\pi x/a)dx$$

$$u(x,y)=.....\#\#\#$$

Ex(15-8) การหาศักย์ไฟฟ้าสถิตในระบบทรงกระบอก (cylindrical coordinates)



Laplace equation: $$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}+\frac{\partial^2 u}{\partial z^2}=0$$

Step1. assume: $$u(r,\phi,z)=R(r)\Phi(\phi)Z(z)$$

Step2. $$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\partial u}{\partial r}\right)=\frac{1}{r}\left[\frac{\partial u}{\partial r}+r\frac{\partial^2 u}{\partial r}\right]=\frac{1}{r}\left[R'\Phi Z+rR''\Phi Z\right],\qquad \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}=\frac{1}{r^2}R\Phi''Z,\qquad \frac{\partial^2 u}{\partial z^2}=R\Phi Z''$$

$$\therefore\quad \frac{1}{r}\left[R'\Phi Z+rR''\Phi Z\right]+\frac{1}{r^2}R\Phi''Z+R\Phi Z''=0$$

multiply by $\frac{r^2}{R\Phi Z}$: $$\frac{r}{R}R'+\frac{r^2}{R}R''+\frac{1}{\Phi}\Phi''+\frac{r^2}{Z}Z''=0\quad\Rightarrow\quad \frac{r}{R}\left(\frac{d}{dr}\right)\left[r\frac{dR}{dr}\right]+\frac{1}{\Phi}\Phi''+\frac{r^2}{Z}Z''=0$$

1) *assume:* $$\Phi''/\Phi=-\nu^2\quad\Rightarrow\quad \Phi''+\nu^2\Phi=0\quad\Rightarrow\quad \boxed{\Phi=B_1\sin(\nu\phi)+B_2\cos(\nu\phi)}$$

2) $$\therefore\quad \frac{r}{R}\left(\frac{d}{dr}\right)\left[r\frac{dR}{dr}\right]-\nu^2+\frac{r^2}{Z}Z''=0\quad\Rightarrow\quad \frac{1}{rR}\left(\frac{d}{dr}\right)\left[r\frac{dR}{dr}\right]-\left(\frac{\nu}{r}\right)^2+\frac{1}{Z}Z''=0$$

assume $$Z''/Z=k^2\quad\Rightarrow\quad \boxed{Z=C_1e^{kz}+C_2e^{-kz}}$$

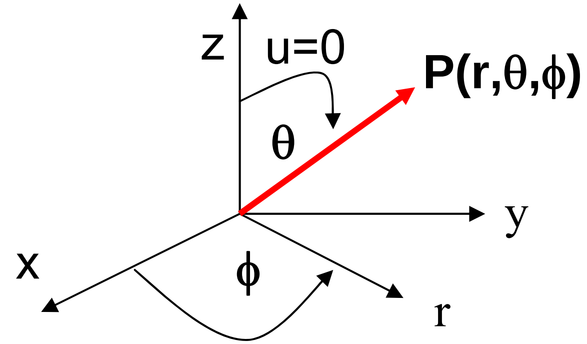
3) $$\therefore\quad \frac{1}{rR}\left(\frac{d}{dr}\right)\left[r\frac{dR}{dr}\right]-\left(\frac{\nu}{r}\right)^2+k^2=0\quad\Rightarrow\quad \frac{1}{r}\left(\frac{d}{dr}\right)\left[r\frac{dR}{dr}\right]+\left[k^2-\left(\frac{\nu}{r}\right)^2\right]R=0$$

$$\Rightarrow\quad \frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left[k^2-\left(\frac{\nu}{r}\right)^2\right]R=0\qquad :\textit{Bessel equation}$$

$$\Rightarrow\quad \boxed{R=A_1J_\nu(kr)+A_2Y_\nu(kr)}$$

$$\therefore\quad \boxed{u(r,\phi,z)=R(r)\Phi(\phi)Z(z)=[A_1J_\nu(kr)+A_2Y_\nu(kr)][B_1\sin(\nu\phi)+B_2\cos(\nu\phi)][C_1e^{kz}+C_2e^{-kz}]}\ ...\#\#\#$$

Ex(15-9) การหาศักย์ไฟฟ้าสถิตในระบบทรงกระบอก (spherical coordinates)



Laplace equation: $\frac{1}{r^2}\frac{\partial}{\partial r}(\frac{r^2\partial u}{\partial r})+\frac{1}{r^2\sin\theta}(\frac{\partial}{\partial\theta})[\sin\theta\frac{\partial u}{\partial\theta}]+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}=0$

Step1. assume: $u(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)$

Step2. $\frac{1}{r^2}\frac{\partial}{\partial r}(\frac{r^2\partial u}{\partial r})=\frac{\Theta\Phi}{r^2}\frac{d}{dr}(\frac{r^2 dR}{dr}),\quad \frac{1}{r^2\sin\theta}(\frac{\partial}{\partial\theta})[\sin\theta\frac{\partial u}{\partial\theta}]=\frac{R\Phi}{r^2\sin\theta}(\frac{d}{d\theta})[\sin\theta\frac{d\Theta}{d\theta}],$

$\therefore\quad \frac{\Theta\Phi}{r^2}\frac{d}{dr}(\frac{r^2 dR}{dr})+\frac{R\Phi}{r^2\sin\theta}(\frac{d}{d\theta})[\sin\theta\frac{d\Theta}{d\theta}]+\frac{R\Theta}{r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=0$

multiply by $\frac{r^2\sin^2\theta}{R\Theta\Phi}$: $\quad \frac{\sin^2\theta}{R}\frac{d}{dr}(\frac{r^2 dR}{dr})+\frac{\sin\theta}{\Theta}(\frac{d}{d\theta})[\sin\theta\frac{d\Theta}{d\theta}]+\frac{\Phi''}{\Phi}=0$

1) *assume:* $\Phi''/\Phi=-n^2\quad\Rightarrow\quad \Phi''+n^2\Phi=0\quad\Rightarrow\quad$ $\Phi=C_1\sin(n\phi)+C_2\cos(n\phi)$

2) $\therefore\quad \frac{\sin^2\theta}{R}\frac{d}{dr}(\frac{r^2 dR}{dr})+\frac{\sin\theta}{\Theta}(\frac{d}{d\theta})[\sin\theta\frac{d\Theta}{d\theta}]-n^2=0\quad\Rightarrow\quad \frac{1}{R}\frac{d}{dr}(\frac{r^2 dR}{dr})=-\frac{1}{\Theta\sin\theta}(\frac{d}{d\theta})[\sin\theta\frac{d\Theta}{d\theta}]+(n/\sin\theta)^2$

assume $\frac{1}{R}\frac{d}{dr}(\frac{r^2 dR}{dr})=m(m+1)\quad\Rightarrow\quad \frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{2r}{R}\frac{dR}{dr}-m(m+1)=0$

multiply by R $\Rightarrow\quad r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-m(m+1)R=0$ *: Euler equation* $\therefore$ $R=A_1r^m+A_2r^{-(m+1)}$

3) $\therefore\quad -\frac{1}{\Theta\sin\theta}(\frac{d}{d\theta})[\sin\theta\frac{d\Theta}{d\theta}]+(n/\sin\theta)^2=m(m+1)\quad\Rightarrow\quad (\frac{d}{d\theta})[\sin\theta\frac{d\Theta}{d\theta}]+[m(m+1)\sin\theta-n^2/\sin\theta]\Theta=0$

assume $v=\cos\theta,\ dv=-\sin\theta d\theta,\ \frac{d}{d\theta}=\frac{dv}{d\theta}\frac{d}{dv}=-\sin\theta\frac{d}{dv}=-(1-v^2)^{1/2}\frac{d}{dv}$

$\Rightarrow\quad (1-v^2)\frac{d^2\Theta}{dv^2}-2v\frac{d\Theta}{dv}+[m(m+1)-\frac{n^2}{1-v^2}]\Theta=0$

if $n=0\quad\Rightarrow\quad (1-v^2)\frac{d^2\Theta}{dv^2}-2v\frac{d\Theta}{dv}+m(m+1)\Theta=0$ *: Legendre equation* $\therefore$ $\Theta=P_m(v)=P_m(\cos\theta)$

if $n\neq 0\quad\Rightarrow\quad \Theta=P_m^n(v)=\{\frac{(1-v^2)^{n/2}}{2m(m!)}\}\left(\frac{d}{dv}\right)^{n+m}(v^2-1)^m$

$\therefore$ $u(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)=[A_1r^m+A_2r^{-(m+1)}][P_m^n(\cos\theta)][C_1\sin(n\phi)+C_2\cos(n\phi)]$...###

$$Ex. \qquad 3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad u(0,t)=0,\ \ u(2,t)=0,\ \ u(x,0)=x,\ \ t\ge 0, 0\le x\le 2$$

$$Step1. \qquad assume: \qquad u(x,t) = X(x)T(t)$$

$$Step2. \qquad \therefore \qquad 3X''T = XT' \qquad \Rightarrow \qquad \frac{X''}{X} = \frac{1}{3}\frac{T'}{T} = const. = -\omega^2$$

$$1) \qquad X''/X = -\omega^2, \qquad \Rightarrow \qquad X'' + \omega^2 X = 0, \qquad X = A\cos(\omega x) + B\sin(\omega x)$$

$$2) \qquad T'/T = -3\omega^2, \qquad \Rightarrow \qquad T' + 3\omega^2 T = 0, \qquad T = Ce^{-3\omega^2 t}$$

$$at \quad x=0, \quad u(0,t) = (A+0)T = 0 \qquad \Rightarrow \qquad A = 0$$

$$at \quad x=2, \quad u(2,t) = [0\cos(2\omega) + B\sin(2\omega)]T = 0, \qquad \Rightarrow \qquad 2\omega = n\pi .....n = 1,2,3...$$

$$\Rightarrow \qquad u(x,t) = B\sin(n\pi x/2)Ce^{-3n^2\pi^2 t/4}$$

$$\therefore \qquad u(x,t) = \sum_{n=1}^{\infty} D_n \sin(n\pi x/2)e^{-3n^2\pi^2 t/4}$$

$$at \quad x=x, \quad u(x,0) = x = \sum_{n=1}^{\infty} D_n \sin(n\pi x/2)$$

$$since \qquad x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{4}{n\pi}\sin(\frac{n\pi x}{2}) \qquad then \qquad D_n = (-1)^{n-1}\frac{4}{n\pi}$$

$$\therefore \qquad u(x,t) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{4}{n\pi}\sin(\frac{n\pi x}{2})e^{-3n^2\pi^2 t/4} ...\#\#\#$$

$$Ex. \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad u(0,t)=0, \quad u(x,0)=f(x)............ f(x)=\begin{cases} 1.........0 \le x \le 1 \\ 0..............x>1 \end{cases}$$

Step1. assume: $u(x,t) = X(x)T(t)$

Step2. $\therefore \quad X''T = XT' \quad \Rightarrow \quad \frac{X''}{X} = \frac{T'}{T} = const. = -\omega^2$

1) $X''/X = -\omega^2, \quad \Rightarrow \quad X'' + \omega^2 X = 0, \quad X = A\cos(\omega x) + B\sin(\omega x)$

2) $T'/T = -\omega^2, \quad \Rightarrow \quad T' + \omega^2 T = 0, \quad T = Ce^{-\omega^2 t}$

at $x = 0, \quad u(0,t) = (A+0)T = 0 \quad \Rightarrow \quad A = 0$

$\Rightarrow \quad u(x,t) = B\sin(\omega x)Ce^{-\omega^2 t} \quad = D\sin(\omega x)e^{-\omega^2 t}$

$\therefore \quad u(x,t) = \int_0^\infty D(\omega)\sin(\omega x)e^{-\omega^2 t}d\omega$

at $x = x, \quad u(x,0) = f(x) = \int_0^\infty D(\omega)\sin(\omega x)d\omega$

$$D(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)dx \quad = \frac{2}{\pi}\int_0^1 (1)\sin(\omega x)dx \quad = \frac{2}{\pi}\frac{[1-\cos(\omega)]}{\omega}$$

$$\therefore \quad u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{[1-\cos(\omega)]}{\omega}\sin(\omega x)e^{-\omega^2 t}d\omega ...\#\#\#$$

$$Ex. \qquad 2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad u(0,t)=0, \; u(4,t)=0, \; u(x,0)=3\sin \pi x - 2\sin 5\pi x \ldots 0 \le x \le 4, \; t \ge 0$$

$$assume: \qquad u(x,t) = X(x)T(t) \quad \therefore \quad 2X''T = XT' \Rightarrow \frac{X''}{X} = \frac{T'}{2T} = const. = -\omega^2$$

$$1) \qquad X''/X = -\omega^2, \qquad \Rightarrow \qquad X'' + \omega^2 X = 0, \qquad X = A\cos(\omega x) + B\sin(\omega x)$$

$$2) \qquad T'/T = -2\omega^2, \qquad \Rightarrow \qquad T' + 2\omega^2 T = 0, \qquad T = Ce^{-2\omega^2 t}$$

$$at \quad x=0, \quad u(0,t) = (A+0)T = 0 \qquad \Rightarrow \qquad A = 0$$

$$at \quad x=4, \quad u(4,t) = [0 + B\sin(4\omega)]T = 0 \qquad \Rightarrow \qquad \omega = n\pi/4 \ldots\ldots n = 1,2,3,\ldots$$

$$\Rightarrow \qquad u(x,t) = B\sin(\omega x)Ce^{-2\omega^2 t} \qquad = D\sin(\omega x)e^{-\omega^2 t}$$

$$\therefore \qquad u(x,t) = \sum_{1}^{\infty} D_n \sin(n\pi x/4)e^{-n^2\pi^2 t/8}$$

$$at \quad x = x, \quad u(x,0) = 3\sin \pi x - 2\sin 5\pi x = \sum_{1}^{\infty} D_n \sin(n\pi x/4)$$

$$\int_0^4 D_4 \sin(n\pi x/4)\sin(4\pi x/4)dx = \int_0^4 [3\sin \pi x - 2\sin 5\pi x]\sin(4\pi x/4)dx \qquad \Rightarrow \qquad D_4 = 3,$$

$$\int_0^4 D_{20} \sin(n\pi x/4)\sin(20\pi x/4)dx = \int_0^4 [3\sin \pi x - 2\sin 5\pi x]\sin(20\pi x/4)dx \qquad \Rightarrow \qquad D_{20} = -2,$$

$$D_n = \frac{1}{\pi}\int_0^{\pi} [3\sin \pi x - 2\sin 5\pi x]\sin(n\pi x/4)dx \qquad = 0$$

$$\therefore \qquad u(x,t) = 3e^{-2\pi^2 t}\sin(4\pi x/4) - 2e^{-50\pi^2 t}\sin(20\pi x/4) = 3e^{-2\pi^2 t}\sin(\pi x) - 2e^{-50\pi^2 t}\sin(5\pi x) \ldots \#\#\#$$

Note: Hyperbolic function

$$e^{jx} = \cos(x) + j\sin(x)$$

$$e^{-jx} = \cos(x) - j\sin(x)$$

$$\sin(x) = \frac{1}{2j}(e^{jx} - e^{-jx})$$

$$\cos(x) = \frac{1}{2}(e^{jx} + e^{-jx})$$

$$e^{x} = \cosh(x) + \sinh(x)$$

$$e^{-x} = \cosh(x) - \sinh(x)$$

$$\sinh(x) = \frac{1}{2}(e^{x} - e^{-x})$$

$$\cosh(x) = \frac{1}{2}(e^{x} + e^{-x})$$

$$\cos^2(x) + \sin^2(x) = 1$$

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$$

$$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$$

$$\frac{d}{dx}\sin(u) = \cos(u)\frac{du}{dx}$$

$$\frac{d}{dx}\cos(u) = -\sin(u)\frac{du}{dx}$$

$$\frac{d}{dx}\sinh(u) = \cosh(u)\frac{du}{dx}$$

$$\frac{d}{dx}\cosh(u) = \sinh(u)\frac{du}{dx}$$

$\Rightarrow$ *multiply* j *by* $-j$

Problem7/360 เพลาหมุนยาว L วางในแกน x จาก x=0~x=L เมื่อมีทอร์กมาหมุนจะบิดเป็นมุม θ ตามสมการคลื่น จงหามุมที่บิดไป θ(x,t) เมื่อ

1. x=0, θ(0,t)=0..........t>0,
2. x=L, [∂θ/∂x]=0...........t>0
3. t=0, θ(x,0)=f(x).......0≤x ≤L,
4. t=0, [∂ θ /∂t] =g(x)...0≤x ≤L,

$$Equation: \quad \frac{\partial^2\theta}{\partial t^2} = a^2\frac{\partial^2\theta}{\partial x^2}, \qquad assume: \quad \theta(x,t) = X(x)T(t)$$

$$\therefore \quad XT'' = a^2X''T \quad \Rightarrow \quad \frac{X''}{X} = \frac{T''}{a^2T} = const.$$

$$case \quad 1: \quad const. = +\omega^2$$

$$1) \quad X''/X = \omega^2, \quad \Rightarrow \quad X'' - \omega^2X = 0, \quad X = A_1e^{\omega x} + B_1e^{-\omega x}$$

$$2) \quad T''/T = (\omega a)^2, \quad \Rightarrow \quad T'' - (\omega a)^2T = 0, \quad T = C_1e^{\omega at} + D_1e^{-\omega at}$$

$$at \quad x = 0, \quad \theta(0,t) = (A_1 + B_1)T = 0 \quad \Rightarrow \quad A_1 = -B_1 \quad \Rightarrow \quad \theta(x,t) = A_1(e^{\omega x} - e^{-\omega x})T$$

$$x = L: \quad \frac{\partial\theta}{\partial x} = 0 = \omega A_1(e^{\omega L} + e^{-\omega L})T \quad \Rightarrow \quad A_1 = B_1 = 0 \quad \therefore \quad const. \neq +\omega^2$$

$$case \quad 2: \quad const. = 0$$

$$1) \quad X''/X = 0, \quad \Rightarrow \quad X'' = 0, \quad X = A_1x + B_1$$

$$2) \quad T''/T = 0, \quad \Rightarrow \quad T'' = 0, \quad T = C_1t + D_1$$

$$at \quad x = 0, \quad \theta(0,t) = (0 + B_1)T = 0 \quad \Rightarrow \quad B_1 = 0 \quad \Rightarrow \quad \theta(x,t) = (A_1x)T$$

$$x = L: \quad \frac{\partial\theta}{\partial x} = 0 = (A_1)T \quad \Rightarrow \quad A_1 = B_1 = 0 \quad \therefore \quad const. \neq 0$$

$case\quad 3: \qquad const. = -\omega^2$

$$1)\qquad X''/X = -\omega^2, \qquad \Rightarrow \qquad X''+\omega^2 X = 0, \qquad X = A_3\cos(\omega x) + B_3\sin(\omega x) \qquad = a_3 e^{j\omega x} + b_3 e^{-j\omega x}$$

$$2)\qquad T''/T = -(a\omega)^2, \qquad \Rightarrow \qquad T''+(a\omega)^2 T = 0, \qquad T = C_3\cos(a\omega t) + D_3\sin(a\omega t) \qquad = c_3 e^{ja\omega t} + d_3 e^{-ja\omega t}$$

$$x = 0, \quad \theta(0,t) = [A_3 + 0]T = 0, \qquad \Rightarrow \qquad A_3 = 0$$

$$\therefore \qquad \theta(x,t) = B_3\sin(\omega x)[C_3\cos(a\omega t) + D_3\sin(a\omega t)] = \sin(\omega x)[h_3\cos(a\omega t) + k_3\sin(a\omega t)]$$

$$x = L, \quad \partial\theta/\partial x\,\Big|_{x=L} = 0 = \omega\cos(\omega L)[h_3\cos(a\omega t) + k_3\sin(a\omega t)] \qquad \Rightarrow \qquad \omega = r\pi/2L \ldots\ldots r = 1,2,3,\ldots$$

$$\therefore \qquad \theta(x,t) = \sin(r\pi x/2L)[h_3\cos(r\pi at/2L) + k_3\sin(r\pi at/2L)]$$

$$\therefore \qquad \theta(x,t) = \sum_{r=1}^{\infty}\theta(x,t) \qquad = \sum_{r=1}^{\infty} \sin(r\pi x/2L)[h_r\cos(r\pi at/2L) + k_r\sin(r\pi at/2L)]$$

$$t = 0, \quad \theta(x,0) = f(x) = \sum_{r=1}^{\infty} \sin(r\pi x/2L)[h_r(1) + k_r(0)]$$

$$\therefore \qquad \int_0^L f(x_1)\sin(r\pi x_1/2L)dx_1 = \int_0^L h_r\sin(r\pi x/2L)\sin(r\pi x/2L)dx = h_r(L/2) \ldots\#\#\#$$

$$\therefore \qquad \frac{\partial\theta}{\partial t} = \sum_{r=1}^{\infty} \sin(r\pi x/2L)[r\pi a/2L][-h_r\sin(r\pi at/2L) + k_r\cos(r\pi at/2L)]$$

$$t = 0, \quad \left[\frac{\partial\theta}{\partial t}\right]_{t=0} = g(x) \quad = \sum_{r=1}^{\infty} \sin(r\pi x/2L)[r\pi a/2L][0 + k_r]$$

$$\int_0^L g(x_1)\sin(r\pi x_1/2L)dx_1 = \int_0^L \sum_1^{\infty}[r\pi a k_r/2L]\sin(r\pi x_1/2L)\sin(r\pi x_1/2L)dx$$

$$\therefore \qquad [r\pi a k_r/2L](L/2) = \int_0^L g(x_1)\sin(r\pi x_1/2L)dx_1 \quad \Rightarrow \quad k_r = (4/r\pi a)\int_0^L g(x_1)\sin(r\pi x_1/2L)dx_1 \ldots\#\#\#$$

Problem 8/360 เพลาหมุนยาว L วางในแกน x จาก x=0~x=L เมื่อมีทอร์กมาหมุนจะบิดเป็นมุม θ ตามสมการคลื่น จงหามุมที่บิดไป θ(x,t) เมื่อ

1. x=0, ∂θ/∂x=0……….t>0,
2. x=L, [∂θ/∂x]=0………..t>0
3. t=0, θ(x,0)=f(x)…….0≤x ≤L,
4. t=0, [∂ θ /∂t] =g(x)…0≤x ≤L,

$$Equation: \quad \frac{\partial^2\theta}{\partial t^2} = a^2\frac{\partial^2\theta}{\partial x^2}, \qquad assume: \quad \theta(x,t) = X(x)T(t)$$

$$\therefore \quad XT'' = a^2X''T \quad \Rightarrow \quad \frac{X''}{X} = \frac{T''}{a^2T} = const.$$

$$case \;\; 1: \quad const. = +\omega^2$$

$$1) \quad X''/X = \omega^2, \quad \Rightarrow \quad X'' - \omega^2X = 0, \quad X = A_1e^{\omega x} + B_1e^{-\omega x}$$

$$2) \quad T''/T = (\omega a)^2, \quad \Rightarrow \quad T'' - (\omega a)^2T = 0, \quad T = C_1e^{\omega at} + D_1e^{-\omega at}$$

$$at \quad x = 0, \quad \partial\theta/\partial x = 0 = \omega(A_1 - B_1)T = 0 \quad \Rightarrow \quad A_1 = B_1$$

$$\Rightarrow \quad \theta(x,t) = A_1(e^{\omega x} + e^{-\omega x})T$$

$$at \quad x = L, \quad \partial\theta/\partial x = 0 = \omega A_1(e^{\omega L} - e^{-\omega L})T \quad \Rightarrow \quad A_1 = B_1 = 0$$

$$\therefore \quad const. \neq +\omega^2$$

$$Equation: \qquad \frac{\partial^2\theta}{\partial t^2} = a^2 \frac{\partial^2\theta}{\partial x^2}, \qquad assume: \qquad \theta(x,t) = X(x)T(t)$$

$$case \;\; 2: \qquad const. = 0$$

$$1) \qquad X''/X = 0, \qquad \Rightarrow \qquad X'' = 0, \qquad X = A_1 x + B_1$$

$$2) \qquad T''/T = 0, \qquad \Rightarrow \qquad T'' = 0, \qquad T = C_1 t + D_1$$

$$at \quad x = 0, \quad \partial\theta/\partial x = 0 = (A_1)T \qquad \Rightarrow \qquad A_1 = 0$$

$$\Rightarrow \qquad \theta(x,t) = B_1(C_1 t + D_1) = h_1 t + k_1$$

$$at \quad t = 0, \quad \theta(x,0) = f(x) = h_1 \qquad \Rightarrow \qquad h_1 = f(x)$$

$$at \quad t = 0, \quad \partial\theta/\partial t = g(x) = k_1 \qquad \Rightarrow \qquad k_1 = g(x)$$

$case\ \ 3: \qquad const. = -\omega^2$

1) $X''/X = -\omega^2, \quad \Rightarrow \quad X'' + \omega^2 X = 0, \quad X = A_3\cos(\omega x) + B_3\sin(\omega x) \quad = a_3 e^{j\omega x} + b_3 e^{-j\omega x}$

2) $T''/T = -(a\omega)^2, \quad \Rightarrow \quad T'' + (a\omega)^2 T = 0, \quad T = C_3\cos(a\omega t) + D_3\sin(a\omega t) \quad = c_3 e^{ja\omega t} + d_3 e^{-ja\omega t}$

$x = 0, \quad \partial\theta/\partial x = \omega[-A_3\sin(\omega x) + B_3\cos(\omega x)]T = 0, \quad \Rightarrow \quad B_3 = 0$

$\therefore \quad \theta(x,t) = A_3\cos(\omega x)[C_3\cos(a\omega t) + D_3\sin(a\omega t)] = \cos(\omega x)[h_3\cos(a\omega t) + k_3\sin(a\omega t)]$

$x = L, \quad \partial\theta/\partial x\,\big|_{x=L} = 0 = -\omega\sin(\omega L)[h_3\cos(a\omega t) + k_3\sin(a\omega t)] \quad \Rightarrow \quad \omega = r\pi/L......r = 1,2,3,...$

$\therefore \quad \theta(x,t) = \cos(r\pi x/L)[h_3\cos(r\pi at/L) + k_3\sin(r\pi at/L)]$

$$\therefore \quad \theta(x,t) = \sum_{r=1}^{\infty}\theta(x,t) \quad = \sum_{r=1}^{\infty} \cos(r\pi x/L)[h_r\cos(r\pi at/L) + k_r\sin(r\pi at/L)]$$

$$t = 0, \quad \theta(x,0) = f(x) = \sum_{r=1}^{\infty} \cos(r\pi x/L)[h_r(1) + k_r(0)]$$

$$\therefore \quad \int_0^L f(x_1)\sin(r\pi x_1/L)dx_1 = \int_0^L h_r\cos(r\pi x_1/L)\cos(r\pi x_1/L)dx_1 \quad = h_r\int_0^L (1/2)(1+\cos(2r\pi x_1/L)dx = h_r(L/2)...\#\#\#$$

$$\therefore \quad \frac{\partial\theta}{\partial t} = \sum_{r=1}^{\infty} \cos(r\pi x/L)[r\pi a/L][-h_r\sin(r\pi at/L) + k_r\cos(r\pi at/L)]$$

$$t = 0, \quad \left[\frac{\partial\theta}{\partial t}\right]_{t=0} = g(x) \quad = \sum_{r=1}^{\infty} \cos(r\pi x/L)[r\pi a/L][0 + k_r]$$

$$\int_0^L g(x_1)\cos(r\pi x_1/L)dx_1 = \int_0^L \sum_1^{\infty}[r\pi a k_r/L]\cos(r\pi x_1/L)\cos(r\pi x_1/L)dx_1$$

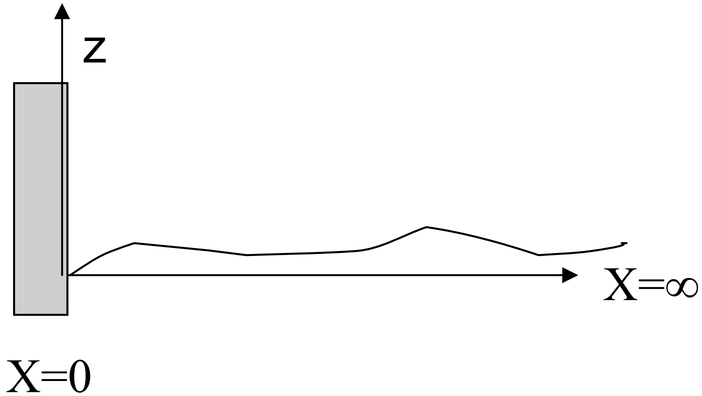
$$\therefore \quad [r\pi a k_r/L](L/2) = \int_0^L g(x_1)\sin(r\pi x_1/2L)dx_1 \quad \Rightarrow \quad k_r = (2/r\pi a)\int_0^L g(x_1)\sin(r\pi x_1/2L)dx_1...\#\#\#$$

$$\Rightarrow \qquad \theta(x,t) = \theta(x,t)\big|_{const.=0} + \theta(x,t)\big|_{const.=-\omega^2}$$

Ex(15-4) การแกว่งของเชือกเป็นตามสมการคลื่น

จงหาการกระจัด z ในเทอมของ x, t เมื่อ

1. $z(0,t)=f(t)$
2. $z(x,t)<\infty$...........$x>0,\ t>0$
3. $z(x,0)=0$..........$0\le x\le\infty,\ t=0$
4. $[\partial z/\partial t]=0$..........$0\le x\le\infty,\ t=0,$



$$\text{wave equation}: \quad \frac{\partial^2 z}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2},$$

$$\text{Laplace transform}:$$

$$L_t[\frac{\partial z(x,t)}{\partial x}]=\int_0^\infty \frac{\partial z(x,t)}{\partial x}e^{-st}dt=\frac{\partial}{\partial x}\left[\int_0^\infty z(x,t)e^{-st}dt\right]=\frac{d}{dx}[L_t[z(x,t)]]$$

$$s^2L_t[z(x,t)]-sz(x,0)-\left[\frac{\partial z}{\partial t}\right]_{t=0}=c^2L_t\left[\frac{\partial^2 z}{\partial x^2}\right]=c^2\frac{d^2}{dx^2}L_t[z(x,t)]$$

$$\because \quad z(x,0)=0, \quad \left[\frac{\partial z}{\partial t}\right]_{t=0}=0$$

$$\therefore \quad \frac{d^2}{dx^2}L_t[z(x,t)]-\left(\frac{s}{c}\right)^2L_t[z(x,t)]=0 \quad\Rightarrow\quad L_t[z(x,t)]=A(s)e^{-sx/c}+B(s)e^{sx/c}$$

$$\because \quad z(x,t)<0.....\forall x \qquad \therefore \quad L_t[z(x,t)]=A(s)e^{-sx/c}$$

$$x=0: \quad L_t[z(0,t)]=L_t[f(t)]=A(s)$$

$$\therefore \quad L_t[z(x,t)]=A(s)e^{-sx/c}=e^{-sx/c}L_t[f(t)]$$

$$\Rightarrow \quad z(x,t)=f(t-x/c)u(t-x/c)...\#\#\#$$