สมการเชิงอนุพันธ์สามัญอันดับที่ 2 (2nd order ODE)

สมการเชิงอนุพันธ์สามัญ	รูปแบบ
Linear 1st ODE	$\frac{dy}{dx}+P(x)y=Q(x)$ $y'+P(x)y=Q(x)$
Linear & homogeneous 2nd ODE	$y''+P(x)y'+Q(x)y=0$
Linear & nonhomogeneous 2nd ODE	$y''+P(x)y'+Q(x)y=R(x)$

ทฤษฎีบทที่ 1

ถ้า y_1 และ y_2 เป็นผลเฉลยใดๆ ของสมการเชิงอนุพันธ์เอกพันธุ์แล้ว **y_3 ที่เป็นผลบวกเชิงเส้น(linear combination)** ของ y_1 และ y_2 จะเป็นผลเฉลยของสมการเชิงอนุพันธ์เอกพันธุ์ดังกล่าวนี้ด้วย

$$y''+P(x)y'+Q(x)y=0$$

$$if \quad y_1, \quad y_2 \quad are \quad solution \quad then$$

$$y_3 = c_1y_1 + c_2y_2 \quad \Rightarrow \quad solution.$$

Proof :

$$\begin{aligned}
y_3'' + P(x)y_3' + Q(x)y_3 &= (c_1y_1 + c_2y_2)'' + P(x)(c_1y_1 + c_2y_2)' + Q(x)(c_1y_1 + c_2y_2) \\
&= (c_1y_1'' + c_2y_2'') + P(x)(c_1y_1' + c_2y_2') + Q(x)(c_1y_1 + c_2y_2) \\
&= c_1(y_1'' + P(x)y_1' + Q(x)y_1) + c_2(y_2'' + P(x)y_2' + Q(x)y_2) \\
&= c_1(0) + c_2(0) = 0....\#\#
\end{aligned}$$

2nd ODE แบบเชิงเส้น (linear form)

$Ex(3-1):\quad y''-4y=0 \;\; has \;\; 2 \;\; particular \;\; solutions, \;\; y_1=e^{2x} \;\; and \;\; y_2=e^{-2x}.$

$Proof \;\; that \;\; y_3=\cosh(2x) \;\; and \;\; y_4=\sinh(2x) \;\; are \;\; also \;\; solutions \;\; of$
$y''-4y=0.$

$$Solution: \quad y_3=c_1y_1+c_2y_2=c_1e^{2x}+c_2e^{-2x}$$
$$y_4=C_1y_1+C_2y_2=C_1e^{2x}+C_2e^{-2x} \qquad are \;\; also \;\; solutions.$$

$$if \quad c_1=c_2=\frac{1}{2} \;\; and \;\; C_1=\frac{1}{2}, \;\; C_2=-\frac{1}{2}$$

then

$$y_3=\frac{(e^{2x}+e^{-2x})}{2}=\cosh(2x)$$
$$y_4=\frac{(e^{2x}-e^{-2x})}{2}=\sinh(2x) \qquad are \;\; also \;\; solutions.$$

ทฤษฎีบทที่ 2

ถ้า y_1 และ y_2 เป็นผลเฉลยของ homogeneous differential eq.

$$y'' + P(x)y' + Q(x)y = 0$$

และถ้านิยาม Wronski determinant หรือ Wronskian ของ y_1 และ y_2 เป็น

$$W(y_1, y_2) \quad = \quad y_1 y_2' - y_1' y_2 \quad = \quad \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

จะปรากฏว่า

1. $W(y_1, y_2) \ = \ 0 \qquad \forall x \in [x_1, x_2] \ \ or$

   $\qquad\qquad\quad \neq \ 0 \qquad \forall x \in [x_1, x_2]$

2. y_1, y_2 *are linearly dependent.* $\Leftrightarrow \quad W(y_1, y_2) = 0$
3. y_1, y_2 *are independent.* *then* $y_3 = c_1 y_1 + c_2 y_2$ *is also solution.*

   c_1, c_2 *are defined constants.*

พิสูจน์ 1 :   ทฤษฎีบทที่ 2

$$\frac{dW}{dx} = W' = y_1'y_2' + y_1 y_2'' - y_1''y_2 - y_1'y_2' = y_1 y_2'' - y_1''y_2$$

since y_1, y_2 *are solutions then*

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0$$

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0$$

$$\Rightarrow \quad [y_1y_2'' - y_1''y_2] + P(x)[y_1y_2' - y_1'y_2] = 0$$

$$\Rightarrow \quad W' + P(x)W = 0 \quad : \quad \text{1ODE}$$

$$\therefore \quad W(y_1, y_2) = C_{12}\exp[-\int P(x)dx] \qquad C_{12} : constant$$

$$if \quad C_{12} = 0 \quad then \quad W(y_1, y_2) = 0 \quad \forall x \in [x_1, x_2]$$

$$if \quad C_{12} \neq 0 \quad then \quad W(y_1, y_2) \neq 0 \quad \forall x \in [x_1, x_2]$$

$$\int P(x)dx] \neq \infty$$

Definition

$$Quantity: \qquad Q_1, Q_2, \ldots Q_n$$

$$c_1 Q_1 + c_2 Q_2 + \ldots + c_n Q_n = 0$$

$$if \qquad c_1 Q_1 + c_2 Q_2 + \ldots + c_n Q_n = 0$$

$$when \quad \forall c_i = 0 \quad (c_1 = c_2 = \ldots = c_n = 0)$$

$$\Rightarrow \quad Q_1, Q_2, \ldots Q_n \quad are \quad independent.$$

$$if \qquad c_1 Q_1 + c_2 Q_2 + \ldots + c_n Q_n = 0$$

$$when \quad \exists c_i \neq 0$$

$$\Rightarrow \quad Q_1, Q_2, \ldots Q_n \quad are \quad dependent.$$

พิสูจน์ 2 : ทฤษฎีบทที่ 2

y_1, y_2 *are linearly dependent.* $\Leftrightarrow W(y_1, y_2) = 0$

1: *if* y_1, y_2 *are dependent then* $y_2 = cy_1$

$$\Rightarrow \quad W(y_1, y_2) = y_1 y_2' - y_1' y_2 = y_1(c\,y_1') - y_1'(cy_1) = 0.....\#\#$$

2. *assume* $c_1 y_1 + c_2 y_2 = 0 \quad \Rightarrow \frac{d}{dx} \Rightarrow \quad c_1 y_1' + c_2 y_2' = 0$

case : *if* y_1 *or* $y_2 = 0 \quad (ex : y_2 = 0)$

$c_1 y_1 = 0$ *and* c_2 *not necessary* $= 0 \Rightarrow \therefore \; y_1, y_2$ *are dependent.###*

if y_1 *or* $y_2 \neq 0$ *then* $c_2 = -c_1 (y_1 / y_2)$

From $c_1 y_1' + c_2 y_2' = 0 \quad \Rightarrow \quad c_1 [y_1' y_2 - y_1 y_2'] / y_2 = 0$

$\Rightarrow \quad -c_1 W(y_1, y_2) = 0 \quad \therefore \quad W(y_1, y_2) = 0$ *while* $c_1 \neq 0$

$\therefore \; y_1, y_2$ *are dependent.###*

y_1, y_2 *are independent.* $\Leftrightarrow W(y_1, y_2) \neq 0$

พิสูจน์ 3 : ทฤษฎีบทที่ 2

y_1, y_2 *are independent.* $\Leftrightarrow$ $y_3 = c_1 y_1 + c_2 y_2$

$$Assume: \quad y_1, y_2, y_3 \quad are \quad solutions \quad of \quad y'' + P(x)y' + Q(x)y = 0$$

$$\Rightarrow \quad W(y_3, y_1) = y_3 y_1' - y_3' y_1 = C_{31} \exp[-\int P(x)dx]$$

$$\Rightarrow \quad W(y_3, y_2) = y_3 y_2' - y_3' y_2 = C_{32} \exp[-\int P(x)dx]$$

$$\therefore \quad y_3 = \frac{(C_{32} y_1 - C_{31} y_2)\exp[-\int P(x)dx]}{(y_1 y_2' - y_1' y_2)}$$

$$since \quad y_1, y_2 \quad are \quad independent \quad then \quad W(y_1, y_2) = (y_1 y_2' - y_1' y_2) \neq 0$$

$$\therefore \quad y_3 = \frac{(C_{32} y_1 - C_{31} y_2)\exp[-\int P(x)dx]}{W(y_1, y_2)} = \frac{(C_{32} y_1 - C_{31} y_2)\exp[-\int P(x)dx]}{C_{12}\exp[-\int P(x)dx]}$$

$$= \frac{C_{32}}{C_{12}} y_1 + \frac{-C_{31}}{C_{12}} y_1 = c_1 y_1 + c_2 y_2 \quad \#\#\#$$

y_3 : *Complete solution or General solution,*

y_1, y_2 : *basis or fundamental system*

ทฤษฎีบทที่ 2

ถ้า y_1 และ y_2 เป็นผลเฉลยของ homogeneous differential eq.

$$y''+P(x)y'+Q(x)y=0$$

จะปรากฏว่า

1. $W(y_1,y_2) = 0 \quad \forall x\in[x_1, x_2]$ *or*
   $\neq 0 \quad \forall x\in[x_1, x_2]$
2. y_1,y_2 *are linearly dependent.* $\Leftrightarrow \quad W(y_1,y_2) = 0$
3. y_1,y_2 *are independent.* *then* $y_3=c_1y_1+c_2y_2$ *is also solution.*
   c_1,c_2 *are defined constants.*

$\Rightarrow$ 2. *and* 3. *can not apply for Non-homogeneous eq.*

Example : 2nd ODE

$Ex(3-2): \quad y''-4y=0 \;\; has \;\; many \;\; solutions \;\; such \;\; as;$

$Solutions \;\; are \;\; y_1 = e^{2x}, \;\; y_2 = e^{-2x}, \;\; y_3 = \cosh(2x), \;\; y_4 = \sinh(2x)$

1. $Using \;\; Wronskian, \;\; show \;\; that \;\; y_i, y_j \;\; are \;\; independent.$
2. $assume \;\; y_5 \;=\; e^{2x} + 2e^{-2x}$

$Show \;\; that \;\; y_5 \;\; is \;\; linear \;\; summation \;\; of \;\; y_1, y_2 \;\; and \;\; y_1, y_2.$

$$A1. \;\; W(y_1,y_2) = y_1 y_2' - y_1' y_2 = e^{2x}(-2e^{-2x}) - (2e^{2x})e^{-2x} = -2-2 = -4 \neq 0 ...\#\#$$

$$W(y_1,y_3) = y_1 y_3' - y_1' y_3 = e^{2x}[2\sinh(2x)] - (2e^{2x})[\cosh(2x)]$$

$$= e^{2x}[(e^{2x}-e^{-2x}) - (e^{2x}+e^{-2x})] \;=\; e^{2x}(-2e^{-2x}) \;= -2 \;\; \neq 0 ...\#\#$$

$$W(y_1,y_4) = y_1 y_4' - y_1' y_4 = e^{2x}[2\cosh(2x)] - (2e^{2x})[\sinh(2x)]$$

$$= e^{2x}[(e^{2x}+e^{-2x}) - (e^{2x}-e^{-2x})] \;=\; e^{2x}(2e^{-2x}) \;= 2 \;\; \neq 0 ...\#$$

$$A2: \quad y_5 \;=\; e^{2x} + 2e^{-2x} \;= y_1 + 2y_2 .\#\#\#$$

$$y_5 \;=\; e^{2x} + 2e^{-2x} \;= [\cosh(2x) + \sinh(2x)] - 2[\cosh(2x) - \sinh(2x)]$$

$$= \; -\cosh(2x) + 3\sinh(2x) \;=\; -y_3 + 3y_4 ...\#\#$$

If y_1 is a solution of $y''+P(x)y'+Q(x)y=0$
then y_2 which is a solution and independent to y_1 could be determined.

Assume: $y(x)=\phi(x)y_1(x)$ is a solution.

$$\Rightarrow \quad y'=\phi y_1' + \phi' y_1 \quad and \quad y''=\phi y_1'' + 2\phi' y_1' + \phi'' y_1$$

From $\quad y'' + P(x)y' + Q(x)y = 0$

$$\Rightarrow \quad (\phi y_1'' + 2\phi' y_1' + \phi'' y_1) + P(x)(\phi y_1' + \phi' y_1) + Q(x)\phi y_1 = 0$$

$$[y_1'' + P(x)y_1' + Q(x)y_1]\phi + [2y_1' + P(x)y_1]\phi' + y_1\phi'' = 0$$

$$[2y_1' + P(x)y_1]\phi' + y_1\phi'' = 0$$

$$\phi'' = \frac{d\phi'}{dx} \quad \Rightarrow \quad \frac{d\phi'}{\phi'} + \frac{2dy_1'}{y_1} + P(x)dx = 0$$

$$\int \;\Rightarrow\; \ell n|\phi'| + \ell n|(y_1)^2| + \int P(x)dx = \ell n(C) \;\Rightarrow\; \ell n\left|\frac{\phi'(y_1)^2}{C}\right| + \int P(x)dx = 0$$

$$\therefore \quad \phi' = \frac{C}{(y_1)^2}e^{-\int P(x)dx} \quad \Rightarrow \quad \phi = \int\left(\frac{C}{(y_1)^2}e^{-\int P(x)dx}\right)dx + k$$

$$\therefore \quad y(x) = \phi(x)y_1(x) = \left[\int\left(\frac{C}{(y_1)^2}e^{-\int P(x)dx}\right)dx\right]y_1(x) + ky_1(x) = Cy_2(x) + ky_1(x)$$

$$\Rightarrow \text{another soluton} \quad y_2(x) = \left[\int\left(\frac{e^{-\int P(x)dx}}{[y_1(x)]^2}\right)dx\right]y_1(x) \quad ...\#\#\#$$

$If \quad y_1 \quad is \quad a \quad solution \quad of \quad y''+P(x)y'+Q(x)y=0$

$then \quad y_2 \quad which \quad is \quad a \quad solution \quad and \quad independent \quad to \quad y_1 \quad could \quad be \quad determined.$

$$Assume: \quad y(x)=\phi(x)y_1(x) \quad is \quad a \quad solution.$$

$$\Rightarrow \quad y'=\phi y_1' + \phi' y_1 \quad and \quad y''=\phi y_1'' + 2\phi' y_1' + \phi'' y_1$$

$$From \quad y'' + P(x)y' + Q(x)y = 0 \quad \longrightarrow \quad f(x)$$

$$\Rightarrow \quad (\phi y_1'' + 2\phi' y_1' + \phi'' y_1) + P(x)(\phi y_1' + \phi' y_1) + Q(x)\phi y_1 = 0$$

$$[y_1'' + P(x)y_1' + Q(x)y_1]\phi + [2y_1' + P(x)y_1]\phi' + y_1\phi'' = 0$$

$$[2y_1' + P(x)y_1]\phi' + y_1\phi'' = 0$$

$$\phi'' = \frac{d\phi'}{dx} \quad \Rightarrow \quad \frac{d\phi'}{\phi'} + \frac{2dy_1'}{y_1} + P(x)dx = 0$$

$$\int \Rightarrow \ell n|\phi'| + \ell n|(y_1)^2| + \int P(x)dx = C \quad \Rightarrow \quad \ell n\left|\frac{\phi'(y_1)^2}{C}\right| + \int P(x)dx = 0 \quad \rightarrow \quad \ell n\left|\frac{\phi'(y_1)^2}{C}\right| + \int [P(x)+f(x)]dx = 0$$

$$\therefore \quad \phi' = \frac{C}{(y_1)^2}e^{-\int P(x)dx} \quad \Rightarrow \quad \phi = \int\left(\frac{C}{(y_1)^2}e^{-\int P(x)dx}\right)dx + k \quad \rightarrow \quad \phi = \int\left(\frac{C}{(y_1)^2}e^{-\int [P(x)+f(x)]dx}\right)dx + k$$

$$\therefore \quad y(x) = \phi(x)y_1(x) = \left[\int\left(\frac{C}{(y_1)^2}e^{-\int P(x)dx}\right)dx\right]y_1(x) + ky_1(x) = Cy_2(x) + ky_1(x)$$

$$\Rightarrow another \quad soluton \quad y_2(x) = \left[\int\left(\frac{e^{-\int P(x)dx}}{[y_1(x)]^2}\right)dx\right]y_1(x) \quad ...\#\#\# \quad \rightarrow \quad \left[\int\left(\frac{e^{-\int [P(x)+f(x)]dx}}{[y_1(x)]^2}\right)dx\right]y_1(x)$$

$Ex(3-3):$ *Find complete solution of* $x^2y'' + xy' - y = 0$ *while* $y_1(x) = x$ *is one solution.*

Solution1: *complete solution* $y(x) = \phi(x)y_1(x) = \phi(x)x \Rightarrow y' = \phi + \phi' x \;,\; y'' = \phi' + (\phi' + \phi'' x) = 2\phi' + \phi'' x$

$$\Rightarrow \quad x^2(2\phi' + \phi'' x) + x(\phi + \phi' x) - x\phi = 0$$

$$\Rightarrow \quad \phi'' x + 3\phi' = 0 \quad \Rightarrow \quad \frac{d\phi'}{\phi'} + \frac{3dx}{x} = 0 \quad \Rightarrow \int \Rightarrow \quad \ell n|\phi'| + 3\ell n|x| = \ell n|C| \quad \Rightarrow \quad \ell n|\phi' x^3| = \ell n|C|$$

$$\Rightarrow \quad \phi' = \frac{C}{x^3} \quad \Rightarrow \quad \int \quad \Rightarrow \quad \phi = \frac{-C}{2x^2} + k$$

$$\Rightarrow \quad y = (\frac{-C}{2x^2} + k)y_1 = -\frac{C}{2x} + kx ...\#\#\#$$

Solution2: $x^2y'' + xy' - y = 0 \quad \Rightarrow \quad y'' + \frac{y'}{x} - \frac{y}{x^2} = 0$

$$\therefore \quad P(x) = \frac{1}{x}, \quad Q(x) = \frac{1}{x^2} \quad \Rightarrow \quad \int P(x)dx = \int \frac{dx}{x} = \ell n|x| \quad \Rightarrow \quad e^{-\int P(x)dx} = e^{-\ell n|x|} = \frac{1}{x}$$

$$\therefore \quad y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx = x \int \frac{(1/x)}{(1/x^2)} dx = x \int \frac{1}{x^3} dx = -\frac{1}{2x}$$

$$\therefore \quad y = c_1y_1 + c_2y_2 = c_1x + c_2(-\frac{1}{2x}) = c_1x - \frac{C}{x} ...\#\#\#$$

ทฤษฎีบทที่ 3

If $c_1y_1+c_2y_2$ is solution of **homogeneous** eq. **y''+P(x)y'+Q(x)y=0**
and
Y is a solution of **nonhomogeneous** eq. y''+P(x)y'+Q(x)y=R(x)
then
y= **$c_1y_1+c_2y_2$+Y is complete solution of** y''+P(x)y'+Q(x)y=R(x)

Proof: *assume y and Y are solution of* $y'' + P(x)y' + Q(x)y = R(x)$

then

$$y'' + P(x)y' + Q(x)y = R(x)$$

$$Y'' + P(x)Y' + Q(x)Y = R(x)$$

$$\therefore \quad (y-Y)'' + P(x)(y-Y)' + Q(x)(y-Y) = 0$$

$$\Rightarrow \quad (y-Y) = c_1y_1 + c_2y_2 \quad \textit{is solution of} \quad y'' + P(x)y' + Q(x)y = 0 ...\#\#\#$$

$c_1y_1+c_2y_2$: complementary function of **homogeneous eq.**
Y : particular integral of **nonhomogeneous eq.**

Finding complete solution

Step 1: Find complementary function of homogeneous eq.

$y_h = c_1y_1+c_2y_2$: $y_h''+P(x)y_h'+Q(x)y_h = 0$

(Find y_1 then $y_h=\phi y_1= c_1y_1+c_2y_2$,)

$$\phi = \int\left(\frac{C}{(y_1)^2}e^{-\int[P(x)+f(x)]dx}\right)dx + k$$

Step 2: Find particular integral of nonhomogeneous eq.

$y_p = Y$: $y_p''+P(x)y_p'+Q(x)y_p = R(x)$

Step 3: Complete solution of nonhomogeneous eq.

$y = y_h+y_p$: $y''+P(x)y'+Q(x)y=R(x)$

= c1y1+c2y2+Y

Constant coefficient homogeneous linear equation

2 ODE **homogeneous** eq. : **y''+P(x)y'+Q(x)y=0**

If P(x), Q(x) = const. $\Rightarrow$ ay''+by'+cy=0

Operator: D $\equiv$ d/dx $\Rightarrow$ Dy = dy/dx=y',

D^2y = D(Dy) = (d/dx)(dy/dx) = y''

$\therefore$ ay''+by'+cy = 0 $\Rightarrow$ $(aD^2+bD+c)y = 0$

Solution $y = e^{mx}$ $\Rightarrow$ $(am^2+bm+c)e^{mx} = 0$

since $y = e^{mx} \neq 0$ $\therefore$ $(am^2+bm+c) = 0$:characteristics eq.

$$m_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad , \quad m_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad y_1 = e^{m_1 x} \quad , \quad y_2 = e^{m_2 x}$$

$$ay'' + by' + cy = 0 \quad \Rightarrow \quad am^2 + bm + c = 0$$

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad , \quad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad y_1 = e^{m_1 x} \quad , \quad y_2 = e^{m_2 x}$$

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = e^{m_1 x}(m_2 e^{m_2 x}) - (m_1 e^{m_1 x})e^{m_2 x} = (m_2 - m_1)e^{(m_1 + m_2)x}$$

$$if \quad (m_1 = m_2) \quad then \quad W(y_1, y_2) = 0 \quad \Rightarrow \quad y_1, y_2 \quad are \quad dependent!!!$$

$$Complete \quad solution \quad :$$

$$case \quad 1: \quad b^2 - 4ac > 0 \quad \Rightarrow \quad m_1 \neq m_2 : real \quad \Rightarrow \quad y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$case \quad 2: \quad b^2 - 4ac < 0 \quad \Rightarrow \quad m_1 \neq m_2 : complex$$

$$assume \quad m_1 = p + jq \quad and \quad m_1 = p - jq \quad \Rightarrow \quad y = e^{px}[c_1 e^{qx} + c_2 e^{-qx}]$$

$$\Rightarrow \quad y = e^{px}\{c_1[\cos(qx) + j\sin(qx)] + c_2[\cos(qx) - j\sin(qx)]\}$$

$$= e^{px}\{(c_1 + c_2)\cos(qx) + j(c_1 - c_2)\sin(qx)\}$$

$$= e^{px}\{A\cos(qx) + B\sin(qx)\}$$

$$W(y_1,y_2)=y_1y_2'-y_1'y_2=e^{m_1x}(m_2e^{m_2x})-(m_1e^{m_1x})e^{m_2x}=(m_2-m_1)e^{(m_1+m_2)x}$$

$$if \quad (m_1=m_2) \quad then \quad W(y_1,y_2)=0 \qquad \Rightarrow \qquad y_1,y_2 \quad are \quad dependent!!!$$

Complete solution :

$$case \;\; 3: \qquad b^2-4ac=0 \qquad \Rightarrow \qquad m_1=m_2=-b/(2a)$$

$$\Rightarrow \qquad W(y_1,y_2)=0 \qquad \Rightarrow \qquad y_1,y_2 \quad are \quad dependent$$

$$assume \qquad y_1=e^{m_1x} \qquad \Rightarrow \qquad y=\phi y_1$$

$$y''+P(x)y'+Q(x)y=0 \qquad \Rightarrow \qquad y''+(b/a)y'+(c/a)y=0$$

$$\Rightarrow \qquad \phi=Cy_1\int\frac{1}{(y_1)^2}e^{-\int P(x)dx}dx+ky_1 \qquad =Ce^{m_1x}\int\frac{1}{(e^{m_1x})^2}e^{-\int\frac{b}{a}dx}dx+ky_1$$

$$=Ce^{m_1x}\int\frac{1}{(e^{m_1x})^2}e^{-\int -2m_1dx}dx+ke^{m_1x} \qquad =Ce^{m_1x}\int\frac{1}{(e^{m_1x})^2}e^{2m_1x}dx+ke^{m_1x}$$

$$=Ce^{m_1x}(x)+ke^{m_1x} \qquad =(Cx+k)e^{m_1x}$$

$$\Rightarrow \qquad y_2=xe^{m_1x}$$

Examples

$$Ex1: \quad y'' + 5y' + 6y = 0 \quad \Rightarrow \quad m^2 + 5m + 6 = 0 \quad \Rightarrow \quad m_1 = -2, \quad m_2 = -3$$

$$y = c_1 e^{-2x} + c_2 e^{-3x} \qquad \#\#\#$$

$$Ex2: \quad y'' + 2y' + 10y = 0 \quad \Rightarrow \quad m^2 + 2m + 10 = 0 \quad \Rightarrow \quad m_1 = -1 + j3, \quad m_2 = -1 - j3$$

$$y = c_1 e^{(-1+j3)x} + c_2 e^{(-1-j3)x} = e^{-x}[A\cos(3x) + B\sin(3x)] \qquad \#\#\#$$

$$Ex3: \quad y'' + 4y' + 4y = 0 \quad \Rightarrow \quad m^2 + 4m + 4 = 0 \quad \Rightarrow \quad m_1 = m_2 = -2$$

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$

$$if \quad x = 0, \quad y(0) = 3, \quad y'(0) = 4 \quad then$$

$$y(0) = 3 = c_1(1) + c_2(0)(1) \quad \Rightarrow \quad c_1 = 3$$

$$y'(0) = 4 = c_1(-2)e^{-2x} + c_2 x(-2)e^{-2x} + c_2 e^{-2x} = c_1(-2)(1) + c_2(0)(-2)(1) + c_2(1)$$

$$= -2c_1 + c_2(1) \quad \Rightarrow \quad c_2 = 10$$

$$\therefore \quad y = 3e^{-2x} + 10xe^{-2x} \ldots \#\#\#$$

Particular Integral for Nonhomogeneous Eq.

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y = y_c + y_p \qquad : y_c = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \ldots\ldots complementary \quad function$$

$$: y_p = Y \qquad \ldots\ldots particular \quad integral$$

Method:

1. Method of undetermined coefficients
determined Y from f(x)

2. Method of variation of parameters
determined Y from y_c

1. Method of undetermined coefficients determined Y from f(x)

$$y'' + P(x)y' + Q(x)y = f(x) \qquad : y_p = Y \quad ......particular \quad integral$$

$Condition:$ $Can \; be \; applied \; only \; for$

$f(x) \; which \; is \; linear \; combination \; of \; simple \; function$

$$k(constant), \; x^n (n = integer > 0), \; e^{kx}, \; \cos(kx), \; \sin(kx)$$

$Step1: \; assume \; Y = c_0 f + c_1 f' + c_2 f'' + ... + c_n f^{(n)}$

$Step2: \; calculate \; derivatives \; of \; Y: \; Y', \; Y''$

$Step3: \; replace \; y, \; y', \; y'' \; by \; Y, \; Y', \; Y''$

$determine \; coefficient \; c_i \; by \; comparing \; with \; ODE.$

1. Method of undetermined coefficients determined Y from f(x)

$Ex(3-8): y'' + 4y' + 3y = 30e^{2x}$ $\quad : y_p = Y \quad ......particular \;\; integral$

$\Rightarrow \quad f(x) = 30e^{2x} \quad \Rightarrow \quad (e^{2x})$

$Step1: \;\; assume \;\; Y = c_0 f + c_1 f' + c_2 f'' + ... + c_n f^{(n)}$

$$Y = c_0(e^{2x}) + c_1(2e^{2x}) + c_2(2^2 e^{2x}) + ... + c_n(2^n e^{2x})$$

$\therefore \quad = Ae^{2x}$

$Step2: \;\; calculate \;\; derivatives \;\; of \;\; Y : \; Y', \; Y''$

$$Y' = 2Ae^{2x}, \qquad Y'' = 4Ae^{2x}$$

$Step3: \;\; replace \;\; y, \; y', \; y'' \;\; by \;\; Y, \; Y', \; Y''$

$determine \;\; coefficient \;\; c_i \;\; by \;\; comparing \;\; with \;\; ODE.$

$$(4Ae^{2x}) + 4(2Ae^{2x}) + 3(Ae^{2x}) = 30e^{2x} \quad \Rightarrow \quad 15e^{2x} = 30e^{2x}$$

$\therefore \quad A = 2 \quad \Rightarrow \quad Y = 2e^{2x}$...###

1. Method of undetermined coefficients : determined Y from f(x)

$$Ex(3-9): y'' + 4y' + 3y = 13\sin(2x) \qquad : y_p = Y \quad ......particular \;\; integral$$

$$\Rightarrow \qquad f(x) = 13\sin(2x) \qquad \Rightarrow \qquad \sin(2x)$$

$$Step1: \; assume \;\; Y = c_0 f + c_1 f' + c_2 f'' + ... + c_n f^{(n)}$$

$$Y = c_0(\sin(2x)) + c_1(2\cos(2x)) + c_2(2^2\sin(2x)) + ...$$

$$\therefore \qquad = A\sin(2x) + B\cos(2x)$$

$$Step2: \; calculate \;\; derivatives \;\; of \;\; Y: \; Y', \; Y''$$

$$Y' = 2A\cos(2x) - 2B\sin(2x), \qquad Y'' = -4A\sin(2x) - 4B\cos(2x)$$

$$Step3: \; replace \;\; y, \; y', \; y'' \;\; by \;\; Y, \; Y', \; Y''$$

$$determine \;\; coefficient \;\; c_i \;\; by \;\; comparing \;\; with \;\; ODE.$$

$$[-4A\sin(2x) - 4B\cos(2x)] + 4[2A\cos(2x) - 2B\sin(2x)] + 3[A\sin(2x) + B\cos(2x)] = 13\sin(2x)$$

$$\Rightarrow \qquad (-4A - 8B + 3A)\sin(2x) + (8A - 4B + 3B)\cos(2x) = 13\sin(2x)$$

$$\Rightarrow \qquad (-A - 8B)\sin(2x) + (8A - B)\cos(2x) = 13\sin(2x)$$

$$\therefore (-A - 8B) = 13, (8A - B) = 0 \Rightarrow \; A = -\frac{1}{5}, \; B = -\frac{8}{5}, \; \Rightarrow \; Y = -\frac{1}{5}\sin(2x) - \frac{8}{5}\cos(2x) ...\#\#\#$$

1. Method of undetermined coefficients : determined Y from f(x)

$$Ex(3-10): y'' + 3y' + 2y = 10e^{-3x} + 4x^2 \qquad : y_p = Y \qquad ......particular \quad integral$$

$$\Rightarrow \qquad f(x) = 10e^{-3x} + 4x^2 \qquad \Rightarrow \qquad e^{-3x}, \quad x^2$$

$$Step1: \quad assume \quad Y = c_0 f + c_1 f' + c_2 f'' + ... + c_n f^{(n)}$$

$$Y = c_0 e^{-3x} + c_1(-3)e^{-3x} + c_2(-3)^2 e^{-3x} + ... + C_1x^2 + C_2x + C_3$$

$$\therefore \qquad = c\,e^{-3x} + C_1x^2 + C_2x + C_3$$

$$Step2: \quad calculate \quad derivatives \quad of \quad Y: \quad Y', \quad Y''$$

$$Y' = -3c\,e^{-3x} + 2C_1x + C_2, \qquad Y'' = 9c\,e^{-3x} + 2C_1$$

$$Step3: \quad replace \quad y, \quad y', \quad y'' \quad by \quad Y, \quad Y', \quad Y''$$

$$determine \quad coefficient \quad c_i \quad by \quad comparing \quad with \quad ODE.$$

$$[9c\,e^{-3x} + 2C_1] + 3[-3c\,e^{-3x} + 2C_1x + C_2] + 2[c\,e^{-3x} + C_1x^2 + C_2x + C_3] = 10e^{-3x} + 4x^2$$

$$\Rightarrow \qquad (2c)e^{-3x} + 2C_1x^2 + (6C_1 + 2C_2)x + (2C_1 + 3C_2 + 2C_3) = 10e^{-3x} + 4x^2$$

$$\Rightarrow \qquad c = 5, \qquad C_1 = 2, \qquad C_2 = -6, \qquad C_3 = 7$$

$$\therefore \qquad Y = 5e^{-3x} + 2x^2 - 6x + 7 ...\#\#\#$$

1. Method of undetermined coefficients

Recommended function for particular integral

$$y'' + P(x)y' + Q(x)y = f(x) \qquad : y_p = Y \quad \ldots\ldots particular \quad integral$$

$f(x)$	$Recommended \quad function$
1. $\alpha \quad (constant)$	$A \quad (constant)$
2. $\alpha x^n \quad (n : integer \quad > 0)$	$A_0 + A_1 x + A_2 x^2 + \ldots + A_n x^n$
3. $\alpha e^{\gamma x} \quad (\gamma : complex)$	$Ae^{\gamma x}$
4. $\alpha \cos(kx), \quad \alpha \sin(kx)$	$A\cos(kx) + B\sin(kx)$
5. $\alpha x^n e^{\gamma x} \cos(kx)$ $or \quad \alpha x^n e^{\gamma x} \sin(kx)$	$(A_0 + A_1 x + A_2 x^2 + \ldots + A_n x^n) e^{\gamma x} \cos(kx)$ $+ (B_0 + B_1 x + B_2 x^2 + \ldots + B_n x^n) e^{\gamma x} \sin(kx)$

1. Method of undetermined coefficients : determined Y from f(x)

$$Ex(3-11): y'' + 2y' + y = xe^{-x} + e^{-x} \qquad : y = y_c + y_p \quad ......complete \quad solution$$

$$Step0: \quad (m^2 + 2m + 1) = 0 \quad \Rightarrow \quad m_1 = m_2 = -1 \quad \Rightarrow \therefore \quad y_c = e^{-x} + xe^{-x}$$

$$\Rightarrow \qquad f(x) = xe^{-x} + e^{-x} \quad \approx \quad y_c \quad \therefore \qquad Y = x^2(A_0 + xA_1)e^{-x} = (A_0x^2 + A_1x^3)e^{-x}$$

$$Step1: \quad assume \quad Y = (A_0x^2 + A_1x^3)e^{-x}$$

$$Step2: \quad calculate \quad derivatives \quad of \quad Y: \quad Y', \quad Y''$$

$$Y' = e^{-x}(2A_0x + 3A_1x^2) - e^{-x}(A_0x^2 + A_1x^3) \qquad = e^{-x}[-A_1x^3 + (3A_1 - A_0)x^2 + (2A_0)x]$$

$$Y'' = e^{-x}[-3A_1x^2 + 2(3A_1 - A_0)x + (2A_0)] - e^{-x}[-A_1x^3 + (3A_1 - A_0)x^2 + (2A_0)x]$$

$$= e^{-x}[A_1x^3 - (6A_1 - A_0)x^2 + (6A_1 - 4A_0)x + 2A_0]$$

$$Step3: \quad replace \quad y, \quad y', \quad y'' \quad by \quad Y, \quad Y', \quad Y''$$

determine coefficient c_i *by comparing with ODE.*

$$e^{-x}[A_1x^3 - (6A_1 - A_0)x^2 + (6A_1 - 4A_0)x + 2A_0] \qquad = xe^{-x} + e^{-x}$$

$$+2e^{-x}[-A_1x^3 + (3A_1 - A_0)x^2 + (2A_0)x] + e^{-x}(A_0x^2 + A_1x^3)$$

$$\Rightarrow e^{-x}[(6A_1 - 4A_0 + 4A_0)x + 2A_0] = e^{-x}(6A_1x + 2A_0) \qquad = xe^{-x} + e^{-x}$$

$$\Rightarrow \qquad A_1 = 1/6, \qquad A_0 = 1/2 \qquad \therefore \qquad Y = (\frac{1}{2}x^2 + \frac{1}{6}x^3)e^{-x} ...\#\#\#$$

2. Method of variation of parameters determined Y from y_c

Applicable for all R(x) : $y'' + P(x)y' + Q(x)y = R(x)$

Condition : Must know complementary function

$Let \quad y_1, y_2 \quad are \quad independent \quad solution \quad of \quad y'' + P(x)y' + Q(x)y = 0$

$Assumption 1: \quad Y = u_1 y_1 + u_2 y_2 \qquad u_1, u_2 : \ to \ be \ determined.$

$$\Rightarrow \qquad Y' = (u_1 y_1' + u_1' y_1) + (u_2 y_2' + u_2' y_2)$$

$Assumption 2: \quad u_1' y_1 + u_2' y_2 = 0$

$$\therefore \quad Y' = (u_1 y_1' + u_2 y_2') \quad , \quad Y'' = (u_1 y_1'' + u_1' y_1') + (u_2 y_2'' + u_2' y_2')$$

$$\Rightarrow \quad Y'' + P(x)Y' + Q(x)Y = R(x)$$

$$\therefore \quad [(u_1 y_1'' + u_1' y_1') + (u_2 y_2'' + u_2' y_2')] + P(x)(u_1 y_1' + u_2 y_2') + Q(x)(u_1 y_1 + u_2 y_2) = R(x)$$

$$\Rightarrow \quad u_1[y_1'' + P(x)y_1' + Q(x)y_1] + u_2[y_2'' + P(x)y_2' + Q(x)y_2] + (u_1' y_1' + u_2' y_2') = R(x)$$

$$\Rightarrow \quad (u_1' y_1' + u_2' y_2') = R(x) \quad and \quad u_1' y_1 + u_2' y_2 = 0$$

$$\Rightarrow \quad u_1' = -\frac{y_2 R(x)}{(y_1 y_2' - y_1' y_2)} = -\frac{y_2 R(x)}{W(y_1, y_2)} \quad , \quad u_2' = \frac{y_1 R(x)}{(y_1 y_2' - y_1' y_2)} = \frac{y_1 R(x)}{W(y_1, y_2)}$$

$$Y = u_1 y_1 + u_2 y_2 \quad \Rightarrow Y = y_1 \int -\frac{y_2 R(x)}{W(y_1, y_2)} dx \; + \; y_2 \int \frac{y_1 R(x)}{W(y_1, y_2)} dx ...\#\#\#$$

2. Method of variation of parameters determined Y from y_c

$$Ex(3-12) \quad Find \quad complete \quad solution \quad for \quad y'' + y = \sec(x)$$

$$Solution: \quad characteristics \quad eq. \quad (m^2+1)=0$$

$$\therefore \quad m_1 = j \;,\; m_2 = -j \quad \Rightarrow \quad y_c = e^{0x}[c_1\cos(x) + c_2\sin(x)]$$

$$\Rightarrow \quad y_c = c_1 y_1 + c_2 y_2 \quad \therefore \quad y_1 = \cos(x), \quad y_2 = \sin(x)$$

$$\Rightarrow \quad W(y_1, y_2) = y_1 y_2' - y_1' y_2 = \cos(x)\cos(x) - [-\sin(x)]\sin(x) = 1$$

$$Y = y_1 \int \frac{-y_2 R(x)}{W(y_1,y_2)} dx + y_2 \int \frac{y_1 R(x)}{W(y_1,y_2)} dx$$

$$\Rightarrow \quad = \cos(x) \int \frac{-\sin(x)\sec(x)}{1} dx + \sin(x) \int \frac{\cos(x)\sec(x)}{1} dx$$

$$[\ \int \tan(x)dx = \ln|\sec(x)| \ ,\ \int -\tan(x)dx = -\ln|\sec(x)| = \ln|\cos(x)|]$$

$$= \cos(x)\ln|\cos(x)| + \sin(x)[x]$$

$$\Rightarrow \quad y = y_c + y_p = [c_1\cos(x) + c_2\sin(x)] + [\cos(x)\ln|\cos(x)| + x\sin(x)]...\#\#\#$$

$Euler\ \ equation: \quad x^2y'' + axy' + by = f(x)$

$$\text{Assume}: \quad x = e^t$$

$$\Rightarrow \quad x^2 = e^{2t}, \qquad dx = e^t dt$$

$$\text{from} \quad x^2y'' + axy' + by = f(x)$$

$$\Rightarrow \quad e^{2t}\frac{d}{dx}\left(\frac{dy}{dx}\right) + ae^t\frac{dy}{dx} + by = f(e^t)$$

$$\Rightarrow \quad e^{2t}\frac{d}{e^t dt}\left(\frac{dy}{e^t dt}\right) + ae^t\frac{dy}{e^t dt} + by = f(e^t)$$

$$\Rightarrow \quad e^t\left[\frac{d}{dt}\left(\frac{1}{e^t}\frac{dy}{dt}\right)\right] + a\frac{dy}{dt} + by = f(e^t)$$

$$\Rightarrow \quad e^t\left[\frac{1}{e^t}\frac{d}{dt}\frac{dy}{dt} + \left(-\frac{1}{e^t}\right)\left(\frac{dy}{dt}\right)\right] + a\frac{dy}{dt} + by = f(e^t)$$

$$\Rightarrow \quad \left[\frac{d}{dt}\frac{dy}{dt} - \frac{dy}{dt}\right] + a\frac{dy}{dt} + by = f(e^t)$$

$$\Rightarrow \quad y'' + (a-1)y' + by = f(e^t)$$

Solving Nonhomogeneous 2ODE

$$(aD^2 + bD + c)y = f(x)$$

$$(D - m_1)(D - m_2)y = f(x)/a \quad \Leftarrow \quad am^2 + bm + c = 0$$

$$Assume: \quad (D - m_2)y = u \quad \Leftrightarrow \quad \frac{dy}{dx} - m_2 y = u \quad \ldots\ldots\ldots\ldots\ldots\ldots(*)$$

$$\therefore \quad (D - m_1)u = f(x)/a \quad \Rightarrow \quad \frac{du}{dx} - m_1 u = f(x)/a \quad : ODE1$$

$$\because \quad y' + P(x)y = Q(x) \quad \Rightarrow \quad y = \frac{1}{e^{\int P(x)dx}} \int Q(x) * e^{\int P(x)dx} dx \; + \frac{C}{e^{\int P(x)dx}}$$

$$\therefore \quad u = \frac{1}{e^{\int(-m_1)dx}} \int \left[\frac{f(x)}{a}\right] * e^{\int(-m_1)dx} dx \; + \frac{C}{e^{\int(-m_1)dx}} \quad = e^{m_1 x} \int \left[\frac{f(x)}{a}\right] * e^{-m_1 x} dx + Ce^{m_1 x}$$

$$\therefore \quad y = \frac{1}{e^{\int(-m_2)dx}} \int [u] * e^{\int(-m_2)dx} dx \; + \frac{c}{e^{\int(-m_2)dx}} \quad = e^{m_2 x} \int u * e^{-m_2 x} dx + ce^{m_2 x} \quad ###$$

2. Method of variation of parameters determined Y from y_c

$$Ex(3-11): y'' + 2y' + y = xe^{-x} + e^{-x} \qquad : y = y_c + y_p$$

$$Step0: \quad Find \quad characteristic \quad eq. \qquad (m^2 + 2m + 1) = 0 \quad \Rightarrow \quad m_1 = m_2 = -1$$

$$Step1: \quad Change \quad to \quad 1ODE \quad using \quad operator$$

$$(D - m_1)(D - m_2)y \quad = \quad xe^{-x} + e^{-x}, \qquad select \quad u: \qquad (D - m_2)y = u$$

$$(D+1)(D+1)y \quad = \quad (D+1)u \quad = \quad xe^{-x} + e^{-x} \qquad \Rightarrow \qquad \frac{du}{dx} + u \quad = \quad xe^{-x} + e^{-x}$$

$$\Rightarrow \qquad u = \frac{1}{e^{\int 1dx}} \int (xe^{-x} + e^{-x}) e^{\int 1dx} dx + \frac{C}{e^{\int 1dx}} \qquad = e^{-x} \int (xe^{-x} + e^{-x}) e^{x} dx + Ce^{-x}$$

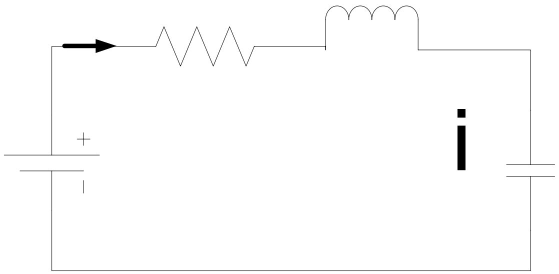
$$= e^{-x} \int (x+1)dx + Ce^{-x} \qquad = e^{-x}(x^2/2 + x) + Ce^{-x}$$

$$Step2: \quad calculate \quad y \quad from \quad u$$

$$(D - m_2)y = u \qquad \Rightarrow \qquad \frac{dy}{dx} + y = u = e^{-x}(\frac{x^2}{2} + x + C)$$

$$y = \frac{1}{e^{\int 1dx}} \int [e^{-x}(\frac{x^2}{2} + x + C)] e^{\int 1dx} dx + \frac{k}{e^{\int 1dx}} \qquad = e^{-x} \int (\frac{x^2}{2} + x + C) dx + ke^{-x}$$

$$= e^{-x}(\frac{x^3}{6} + \frac{x^2}{2} + Cx) + ke^{-x} \qquad = e^{-x}(\frac{x^3}{6} + \frac{x^2}{2}) + Cxe^{-x} + ke^{-x} ...\#\#\#$$



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Ex: Find i(t) while t=0, i(0)=0, Vc(0)=0

$$V_R+V_L+V_C=E,\quad V_R=Ri_R,\quad V_L=L\frac{di_L}{dt},\quad i_C=C\frac{dV_c}{dt},\quad i_R=i_L=i_C=i\quad ,\quad V_C=\frac{1}{C}\int idt+V_{C0}$$

$$Method\quad I:\qquad Ri+L\frac{di}{dt}+\frac{1}{C}\int idt+V_{C0}=E$$

$$\frac{d}{dt}\Rightarrow\quad R\frac{di}{dt}+L\frac{d^2i}{dt^2}+\frac{1}{C}i\;=\;\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{1}{LC}i\;=E\qquad\therefore\quad m^2+\frac{R}{L}m+\frac{1}{LC}=0\;\Rightarrow\;m_1,m_2=-\frac{R}{2L}\pm\sqrt{\frac{R^2}{4L^2}-\left(\frac{1}{LC}\right)^2}$$

$$\Rightarrow\qquad i(t)=C_1e^{m_1t}+C_2e^{m_2t}\qquad :m_1\neq m_2$$

$$t=0,\;\; i(0)=0=C_1+C_2,\;\; and\;\; V_c(0)=0\;\therefore\qquad V_L+V_R+V_C=V_L+Ri(0)+V_C(0)=V_L=L\frac{di}{dt}=E$$

$$\Rightarrow\qquad L(m_1C_1e^{m_1t}+m_2C_2e^{m_2t})=E,\qquad L(m_1C_1+m_2C_2)=E,$$

$$\therefore\qquad C_1=-C_2=\frac{E}{L(m_1-m_2)}=\frac{E}{2L\sqrt{\frac{R^2}{4L^2}-\left(\frac{1}{LC}\right)^2}}=\frac{E}{\sqrt{R^2-\left(\frac{4L}{C}\right)^2}}\qquad\Rightarrow\qquad i(t)=\frac{E}{2L\sqrt{\frac{R^2}{4L^2}-\left(\frac{1}{LC}\right)^2}}(e^{m_1t}-e^{m_2t})..\#\#\#$$

$$\Rightarrow\qquad i(t)=(C_1+C_2t)e^{m_1t}\qquad :m_1=m_2=-\frac{R}{2L}$$

$$t=0,\;\; i(0)=0=C_1,\;\; and\;\; t=0,\;\; V_c(0)=0,\qquad\therefore\;\; L\frac{di}{dt}=L[(C_1+C_2t)m_1+C_2]e^{m_1t}=E\qquad\Rightarrow\qquad C_2=\frac{E}{L}$$

$$\Rightarrow\qquad i(t)=\frac{E}{L}te^{m_1t}=\frac{E}{L}te^{-\frac{R}{2L}t}....\#\#\#$$