สมการเชิงอนุพันธ์ย่อย
(Partial Differential Equation :PDE)

: สมการที่ประกอบด้วย**อนุพันธ์ย่อย อย่างน้อย 1 อนุพันธ์**ของฟังก์ชันที่ประกอบด้วย**ตัวแปรต้นตั้งแต่ 2 ตัวขึ้นไป**

$$4x\frac{\partial z}{\partial x} - 5y\frac{\partial z}{\partial y} = z$$

$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$: 1 *order partial derivative of z by x and y*

$$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$$

$\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$: 2 *order partial derivative of z*

สัญกรณ์(Notation)

$$p = z_x \triangleq \frac{\partial z}{\partial x}$$

$$q \triangleq z_y \triangleq \frac{\partial z}{\partial y}$$

$$r \triangleq z_{xx} \triangleq \frac{\partial^2 z}{\partial x^2}$$

$$s \triangleq z_{xy} \triangleq \frac{\partial^2 z}{\partial x \partial y}$$

$$t \triangleq z_{yy} \triangleq \frac{\partial^2 z}{\partial y^2}$$

PDE	Linear PDE สมการเชิงอนุพันธ์ย่อย เชิงเส้น	Nonlinear PDE สมการเชิงอนุพันธ์ย่อย ไม่เชิงเส้น
ตัวแปรตาม(z)	ฟังก์ชันเชิงเส้น ระดับขั้น1	ฟังก์ชันไม่เชิงเส้น
อนุพันธ์ของตัวแปรตาม (z’, z’’, ..)	ฟังก์ชันเชิงเส้น ระดับขั้น1	ฟังก์ชันไม่เชิงเส้น
	ไม่มีผลคูณระหว่าง z และ z’, z’’,…	มีผลคูณระหว่าง z และ z’, z’’,…
Note: ไม่สนใจว่า x, y จะเป็นอย่างไร	$\frac{\partial z}{\partial x} + x^2 y^3 \frac{\partial^2 z}{\partial y^2} = 0$	$3y^2 \frac{\partial z}{\partial x} + \left(\frac{\partial^2 z}{\partial x^2}\right)^2 + z\frac{\partial z}{\partial y} = z^2$

Order and Degree of PDE

PDE	อันดับ(order) ของสมการ PDE	ระดับขั้น(degree) ของสมการ PDE
	=อันดับสูงสุดของอนุพันธ์	=เลขชี้กำลังสูงสุดของอนุพันธ์อันดับสูงสุด
$3y^2\frac{\partial z}{\partial x}+z\frac{\partial z}{\partial y}=z^2$	1	1
$\frac{\partial z}{\partial x}+x^2y^3\frac{\partial^2 z}{\partial y^2}=0$	2	1
$\frac{\partial z}{\partial y}+x^3y\left(\frac{\partial^2 z}{\partial x^2}\right)^2=0$	2	**2**

ผลเฉลยของ PDE = Integral, primitive

รูปแบบ	ลักษณะของผลเฉลย	Example
ผลเฉลยทั่วไป (general soln.)	ติด**ฟังก์ชัน**ไม่เจาะจง อย่างน้อย 1 ตัว	$xz_x+yz_y=3z$ $Z=x^3\psi(y/x)$
ผลเฉลยบริบูรณ์ (complete soln.)	มี**ค่าคงตัว**ไม่เจาะจง แต่ไม่มี**ฟังก์ชัน**ไม่เจาะจง	$xz_x+yz_y-(zx)^2-(z_y)^2=z$ $z=Ax + By-A^2+B^2$
ผลเฉลยเอกฐาน (singular soln.)	ไม่อาจหาได้จากผลเฉลยบริบูรณ์โดยการแทน**ค่าคงตัวไม่เจาะจง**ด้วยค่าตัวเลข	$xz_x+yz_y-(z_x)^2-z_y)^2=z$ $y=(x^{2+}y^{2)}/4$

การทวนสอบ (Verification)

$$Ex(12-4): \quad Show \quad that \quad z = ax^2 + ay + c$$

$$is \quad a \quad solution \quad of \qquad \frac{\partial z}{\partial x} = 2x\frac{\partial z}{\partial y}$$

$$Answer: \quad \frac{\partial z}{\partial x} = 2ax, \qquad \frac{\partial z}{\partial y} = a$$

$$\therefore \quad LHS: \quad \frac{\partial z}{\partial x} = 2ax$$

$$\therefore \quad RHS: \quad 2x\frac{\partial z}{\partial y} = 2x(a) = LHS \ldots \#\#\#$$

Find PDE from Solution

- Step 1. หา PDE ของตัวแปรตาม
- Step 2. หา PDE ของฟังก์ชันไม่เจาะจง
- Step 3. กำจัด ค่าคงตัวไม่เจาะจง

  กำจัด ฟังก์ชันไม่เจาะจง

$Ex(12-5): \; Find \quad PDE \quad for \quad z = x^3 \psi\left(\frac{y}{x}\right)$

$Ans: \quad Assume \quad u(x,y) = \frac{y}{x}, \qquad \therefore \; \frac{\partial u}{\partial x} = -\frac{y}{x^2}, \quad \frac{\partial u}{\partial y} = \frac{1}{x}$

$\Rightarrow \qquad \psi = \frac{z}{x^3}$

$Step \;\; 1. \qquad \frac{\partial z}{\partial x} = x^3 \frac{\partial \psi}{\partial x} + 3x^2\psi = x^3 \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial x} + 3x^2\psi$

$= x^3 \frac{\partial \psi}{\partial u}\left(-\frac{y}{x^2}\right) + 3x^2\psi = -xy\frac{\partial \psi}{\partial u} + 3x^2\psi$

$Step \;\; 1\&2. \qquad \frac{\partial z}{\partial y} = x^3 \frac{\partial \psi}{\partial y} = x^3 \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial y}$

$= x^3 \frac{\partial \psi}{\partial u}\left(\frac{1}{x}\right) = x^2 \frac{\partial \psi}{\partial u}$

$Step \;\; 3. \qquad \frac{\partial z}{\partial x} = -xy\frac{\partial \psi}{\partial u} + 3x^2\psi = -xy\left(\frac{1}{x^2}\frac{\partial z}{\partial y}\right) + 3x^2\left(\frac{z}{x^3}\right)$

$= -y\left(\frac{1}{x}\frac{\partial z}{\partial y}\right) + 3\left(\frac{z}{x}\right)$

$\therefore \qquad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z \ldots \#\#\#$

$Ex(12-6).\quad Find\quad PDE\quad of \qquad x^2+y^2+(z-c)^2=r^2$

Ans.		$f(x,y,z)=x^2+y^2+(z-c)^2-r^2=0$
Step 1.		$\frac{\partial f}{\partial x}=2x+2(z-c)\frac{\partial z}{\partial x}=0......................(1)$ $\frac{\partial f}{\partial y}=2y+2(z-c)\frac{\partial z}{\partial y}=0....................(2)$
Step 2.	*from* (1) *from* (2)	$2(z-c)=-2x/(\frac{\partial z}{\partial x})=0$ $2y+\frac{-2x}{\frac{\partial z}{\partial x}}\frac{\partial z}{\partial y}=0$
Step 3.	$\Rightarrow$	$2y\frac{\partial z}{\partial x}-2x\frac{\partial z}{\partial y}=0...\#\#\#$

$Ex(12-7).\quad Find\quad PDE\quad for\quad z = ax^2 + by^2 + c$

$Ans.$

$Step\ 1.\quad \frac{\partial z}{\partial x} = 2ax, \qquad \frac{\partial z}{\partial y} = 2by$

$$\frac{\partial^2 z}{\partial x^2} = 2a, \qquad \frac{\partial^2 z}{\partial y^2} = 2b, \qquad \frac{\partial^2 z}{\partial x \partial y} = 0$$

$Step\ 1.\quad \frac{\partial z}{\partial x} = (\frac{\partial^2 z}{\partial x^2})x, \qquad \frac{\partial z}{\partial y} = (\frac{\partial^2 z}{\partial y^2})y, \qquad \frac{\partial^2 z}{\partial x \partial y} = 0$

$Step\ 3.\quad \frac{1}{x}\frac{\partial z}{\partial x} = (\frac{\partial^2 z}{\partial x^2}), \qquad \frac{1}{y}\frac{\partial z}{\partial y} = (\frac{\partial^2 z}{\partial y^2}), \qquad \frac{\partial^2 z}{\partial x \partial y} = 0 \ldots \#\#\#$

Solution of PDE with single variable

Solve as in ODE but change unknown constant to known function

$$Ex(12-8). \quad Find \quad solution \quad for \quad \frac{\partial z}{\partial x} = ax + y + C$$

$$Solve \quad as \quad ODE: \quad \frac{dz}{dx} = ax + y \quad \Rightarrow \quad \frac{dz}{dx} = (ax + y)$$

$$\int \quad \Rightarrow \quad z = \frac{1}{2}ax^2 + yx + C$$

$$\therefore \quad z = \frac{1}{2}ax^2 + yx + C$$

$$Change \quad constant \quad C \quad to \quad function \quad \phi:$$

$$\Rightarrow ....z = \frac{1}{2}ax^2 + yx + \phi(y) \quad :\phi(y): arbitrary \quad function...\#\#\#$$

$Ex(12-9).\quad Find\quad solution\quad for\quad \frac{\partial^2 z}{\partial y^2}+3\frac{\partial z}{\partial y}+2z=x^2+e^{-y}$

$$Solve\ as\ ODE: \qquad \frac{d^2 z}{dy^2}+3\frac{dz}{dy}+2z=x^2+e^{-y}$$

$$Characteristic\ eq.: \qquad (m^2+3m+2)=(m+1)(m+2)=0$$

$$\therefore \qquad m=-1,\ -2$$

$$Complementary\ function: \qquad z_c=Ae^{-y}+Be^{-2y}$$

$$Since\ R(y)=x^2+e^{-y}\ has\ the\ same\ form\ as\ z_c\ then$$

$$Particular\ solution: \qquad z_p=C+Dye^{-y}$$

$$\therefore\ z_p'=D(e^{-y}-ye^{-y}), \qquad z_p''=D(-e^{-y}-e^{-y}+ye^{-y})=D(-2e^{-y}+ye^{-y})$$

$$ODE \Rightarrow \qquad D(-2e^{-y}+ye^{-y})+3D(e^{-y}-ye^{-y})+2(C+Dye^{-y})=x^2+e^{-y}$$

$$\Rightarrow \qquad (-2D+3D)e^{-y}+(D-3D+2D)ye^{-y}+2C=x^2+e^{-y}$$

$$\Rightarrow \qquad De^{-y}+2C=x^2+e^{-y} \qquad \Rightarrow \qquad C=\frac{x^2}{2}, \qquad D=1$$

$$\therefore \qquad z=z_c+z_p=[Ae^{-y}+Be^{-2y}]+(\frac{x^2}{2}+ye^{-y})$$

$$\therefore \qquad z=[\phi(x)e^{-y}+\psi(x)e^{-2y}]+(\frac{x^2}{2}+ye^{-y})...\#\#\#$$

สมการเชิงอนุพันธ์ย่อยอันดับที่ 1 (1st order PDE)

General Form:

$P(x,y,z)\dfrac{\partial z}{\partial x}+Q(x,y,z)\dfrac{\partial z}{\partial y} = R(x,y,z)$
$P(x,y,z)z_x+Q(x,y,z)z_y = R(x,y,z)$
$Pp + Qq = R$

$$p \overset{\Delta}{=} z_x \overset{\Delta}{=} \frac{\partial z}{\partial x}, \qquad q \overset{\Delta}{=} z_y \overset{\Delta}{=} \frac{\partial z}{\partial y}$$

Lagrange system

$$Pp + Qq = R \qquad \text{..........(1)}$$

$$\textit{since} \quad dx\frac{\partial z}{\partial x} + dy\frac{\partial z}{\partial y} = dz \quad \Rightarrow \quad (dx)p + (dy)q = dz \qquad \text{........(2)}$$

$$\frac{(2)}{(1)} \quad \Rightarrow \quad \frac{(dx)p + (dy)q}{Pp + Qq} = \frac{dz}{R}$$

$$\Rightarrow \quad (dx - \frac{dz}{R}P)p + (dy - \frac{dz}{R}Q)q = 0 \qquad \text{.........(3)}$$

if p, q *are independent then coefficient in* $(3) = 0.$

$$\Rightarrow \quad (dx - \frac{dz}{R}P) = 0, \qquad (dy - \frac{dz}{R}Q) = 0$$

$$\therefore \quad \frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} \qquad : \textit{Lagrange system}$$

We will find solution $u(x,y,z), v(x,y,z)$ *for Lagrange system.*

Assume $u(x,y,z)=c, \qquad v(x,y,z)=k$

are solutions of Lagrange system while c and k are constants.

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=K$$

$$\Rightarrow \qquad u_x dx = u_x KP, \qquad u_y dy = u_y KQ, \qquad u_z dz = u_z KP$$

$$\sum \;\Rightarrow \qquad u_x dx + u_y dy + u_z dz = K(u_x P + u_y Q + u_z R)$$

For v, *do same as* u: $\quad [v_x dx + v_y dy + v_z dz = K(v_x P + v_y Q + v_z R)]$

then $$\frac{u_x dx + u_y dy + u_z dz}{Pu_x + Qu_y + Ru_z} = K = \frac{v_x dx + v_y dy + v_z dz}{Pv_x + Qv_y + Rv_z}$$

since $\quad du = u_x dx + u_y dy + u_z dz = dc = 0,$

$\quad dv = v_x dx + v_y dy + v_z dz = dc = 0$

then $\quad Pu_x + Qu_y + Ru_z = 0, \qquad Pv_x + Qv_y + Rv_z = 0$

$$\Rightarrow \qquad u_x \frac{P}{R} + u_y \frac{Q}{R} + u_z = 0, \qquad v_x \frac{P}{R} + v_y \frac{Q}{R} + v_z = 0$$

$\Rightarrow$ *Replace* $\frac{P}{R}, \frac{Q}{R}$ *in* $\frac{P}{R}p + \frac{Q}{R}q = 1 \quad \Rightarrow \quad U_x V_y - U_y V_x = 0$

$$Since \quad U_x V_y = U_y V_x = 0 \quad while$$

$$[\quad U_x = u_x + pu_z, \quad U_y = u_y + qu_z$$

$$V_x = v_x + pv_z, \quad V_y = v_y + qu_z \quad]$$

$$\Leftrightarrow \quad \begin{bmatrix} U_x & V_x \\ U_y & V_y \end{bmatrix} \begin{bmatrix} \phi_u \\ \phi_v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \quad \Phi_x = 0, \quad \Phi_y = 0 \quad : \Phi = \phi(u,v)$$

$Ex(13-1).$ *Find general solution for* $x^2 z_x - xyz_y = y^2$

Ans: *From* $Pp + Qq = R \Rightarrow \dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\Rightarrow \quad P = x^2, \quad Q = -xy, \quad R = y^2 \quad \therefore \quad \frac{dx}{x^2} = \frac{dy}{-xy} = \frac{dz}{y^2}$$

$$\frac{dx}{x^2} = \frac{dy}{-xy} \Rightarrow \frac{dx}{x} + \frac{dy}{y} = 0 \Rightarrow \ln|x| + \ln|y| = \ln|c|$$

$$\Rightarrow xy = c \Rightarrow u(x,y,z) = xy = c$$

$$\frac{dy}{-xy} = \frac{dz}{y^2} \Rightarrow \frac{dy}{-c} = \frac{dz}{y^2} \Rightarrow y^2 dy + cdz = 0$$

$$\Rightarrow \frac{y^3}{3} + xyz = k \Rightarrow v(x,y,z) = \frac{y^3}{3} + xyz = k$$

$\therefore$ *General solution:*

1. $\phi(u,v) = \phi(xy, \dfrac{y^3}{3} + xyz) = 0 ...\#\#\#$

2. $u = \varphi(v) \Rightarrow xy = \varphi(\dfrac{y^3}{3} + xyz) ...\#\#\#$

3. $v = \psi(u) \Rightarrow \dfrac{y^3}{3} + xyz = \psi(xy) \Rightarrow z = \dfrac{1}{xy}\left(\psi(xy) - \dfrac{y^3}{3}\right)$

$$\Rightarrow z = \Psi(xy) - \frac{y^2}{3x} ...\#\#\#$$

$\phi, \varphi, \psi, \Psi$: *arbitrary function.*

$Ex(13-2):\quad Find\quad solution\quad for\quad (cy-bz)z_x+(az-cx)z_y=(bx-ay)$

$Ans:\quad pP+qQ=R \quad\Leftrightarrow\quad \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$\Rightarrow\quad P=cy-bz,\quad Q=az-cx,\quad R=bx-ay$

$\therefore\quad \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}\quad\Rightarrow\quad \frac{dx}{cy-bz}=\frac{dy}{az-cx}=\frac{dz}{bx-ay}$

$Select\quad \lambda_1=a,\ \mu_1=b,\ \nu_1=c;\ \Rightarrow\ \lambda_1P+\mu_1Q+\nu_1R\quad =a(cy-bz)+b(az-cx)+c(bx-ay)=0$

$[Using\quad relations:\quad du=u_xdx+u_ydy+u_zdz=0$

$and\quad \frac{u_xdx+u_ydy+u_zdz}{u_xP+u_yQ+u_zR}=K=\frac{v_xdx+v_ydy+v_zdz}{v_xP+v_yQ+v_zR}\quad]$

$\therefore\quad du=\lambda_1dx+\mu_1dy+\nu_1dz=adx+bdy+cdz=0\quad\Rightarrow\quad u=ax+by+cz=C$

$Select\quad \lambda_2=x,\ \mu_2=y,\ \nu_2=z;\ \Rightarrow\ \lambda_2P+\mu_2Q+\nu_2R\quad =x(cy-bz)+y(az-cx)+z(bx-ay)=0$

$\therefore\quad dv=\lambda_2dx+\mu_2dy+\nu_2dz=xdx+ydy+ydz=0\quad\Rightarrow\quad v=x^2+y^2+z^2=K$

$General\quad solution:$

1. $\phi(u,v)=\phi(ax+by+cz,x^2+y^2+z^2)=0$..............................###

2. $u=\varphi(v)\quad\Rightarrow\quad ax+by+cz=\varphi(x^2+y^2+z^2)$...###

3. $v=\psi(u)\quad\Rightarrow\quad x^2+y^2+z^2=\psi(ax+by+cz)$...###

Non-linear 1PDE

$$f(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}) = f(x,y,z,p,q) = 0 \qquad (1) \qquad Step\ \ 2.$$

Lagrange – Charpit method :

assume : $g(x,y,z,p,q,c) = 0 \qquad (2)$

that exact differential $dz = pdx + qdy \qquad (3) \qquad Step\ \ 3.$

$$\frac{\partial}{\partial x}(f) = f_x + f_z p + f_p p_x + f_q q_x = 0 \qquad (4)$$

$$\frac{\partial}{\partial y}(f) = f_y + f_z q + f_p p_y + f_q q_y = 0 \qquad (5)$$

$$\frac{\partial}{\partial x}(g) = g_x + g_z p + g_p p_x + g_q q_x = 0 \qquad (6)$$

$$\frac{\partial}{\partial y}(g) = g_y + g_z q + g_p p_y + g_q q_y = 0 \qquad (7)$$

$$(-g_p)*(4) \Rightarrow (-g_p)(f_x + f_z p + f_p p_x + f_q q_x) = 0 \qquad (5')$$

$$(-g_q)*(5) \Rightarrow (-g_q)(f_y + f_z q + f_p p_y + f_q q_y) = 0 \qquad (6')$$

$$(f_p)*(6) \Rightarrow (f_p)(g_x + g_z p + g_p p_x + g_q q_x) = 0 \qquad (7')$$

$$(f_q)*(7) \Rightarrow (f_q)(g_y + g_z q + g_p p_y + g_q q_y) = 0 \qquad (8')$$

$$\sum (5) \sim (8) \Rightarrow f_p g_x + f_q g_y - (f_x + pf_z)g_p - (f_y + qf_z)g_q + (f_p p + f_q q)g_z = 0$$

$$\Rightarrow \frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)} = \frac{dz}{(pf_p + qf_q)} \qquad Step\ \ 1.$$

$Ex(13-3):$ Find solution for $p^2 = 2xq$

$Ans: \quad f(x,y,z,p,q) = p^2 - 2xq$

$\Rightarrow \quad f_x = -2q, \quad f_y = 0, \quad f_z = 0, \quad f_p = 2p, \quad f_q = -2x$

since $$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)} = \frac{dz}{(pf_p + qf_q)}$$

then $$\frac{dx}{2p} = \frac{dy}{-2x} = \frac{dp}{2q} = \frac{dq}{0} = \frac{dz}{(2p^2 - 2xq)}$$

$$\frac{dq}{0}: \quad \Rightarrow dq = 0 \quad \Rightarrow \quad q = c \quad \Rightarrow \quad g(x,y,z,p,q,c) = q - c = 0$$

$$p^2 = 2xq \quad \Rightarrow p^2 = 2xc \quad \Rightarrow \quad p = (2cx)^{\frac{1}{2}}$$

since $\quad dz = pdx + qdy$

then $$dz = (2cx)^{\frac{1}{2}} dx + cdy$$

$$\therefore \quad z = (2c)^{\frac{1}{2}} \left(\frac{2}{3}\right) x^{\frac{3}{2}} dx + cy + k \ldots \# \# \#$$

$Ex(13-4):\quad Find\quad solution\quad for\quad q = p^2 x$

$Ans:\quad f(x,y,z,p,q) = q - p^2 x$

$\Rightarrow\quad f_x = -p^2,\quad f_y = 0,\quad f_z = 0,\quad f_p = -2px,\quad f_q = 1$

$$since\quad \frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)} = \frac{dz}{(pf_p + qf_q)}$$

$$then\quad \frac{dx}{-2px} = \frac{dy}{1} = \frac{dp}{p^2} = \frac{dq}{0} = \frac{dz}{(-2p^2x + q)}$$

$$Sol.1.\quad \frac{dq}{0}:\quad \Rightarrow dq = 0\quad \Rightarrow\quad q = c\quad \Rightarrow\quad g(x,y,z,p,q,c) = q - c = 0$$

$$from\quad q = p^2 x\quad \Rightarrow p = \left(\frac{c}{x}\right)^{\frac{1}{2}}$$

$$since\quad dz = pdx + qdy$$

$$then\quad dz = \left(\frac{c}{x}\right)^{\frac{1}{2}} dx + cdy\quad \therefore\quad z = 2(cx)^{\frac{1}{2}} + cy + k ...\#\#\#$$

$$Sol.2.\quad \frac{dy}{1} = \frac{dp}{p^2}\quad \Rightarrow\quad p = \frac{1}{(c-y)}\quad \Rightarrow\quad g(x,y,z,p,q,c) = p - \frac{1}{(c-y)} = 0$$

$$from\quad q = p^2 x\quad \Rightarrow\quad q = \frac{x}{(c-y)^2}$$

$$since\quad dz = pdx + qdy$$

$$then\quad dz = \frac{dx}{(c-y)} + \frac{xdy}{(c-y)^2}\quad \therefore\quad z = \frac{x}{(c-y)} + k ...\#\#\#$$

$Ex(13-5):$ *Relation between static magnetic field* $\vec{H}$ *and current density* $\vec{J}$ *follows Ampere's circuital law:* $\nabla \times \vec{H} = \vec{J}$ $;\nabla = \vec{i}(\frac{\partial}{\partial x}) + \vec{j}(\frac{\partial}{\partial y}) + \vec{k}(\frac{\partial}{\partial z})$

$$\vec{H} = H_x\vec{i} + H_y\vec{j} + H_z\vec{k} \qquad \vec{J} = J_x\vec{i} + J_x\vec{j} + J_z\vec{k}$$

if $\vec{J} = -10\vec{j} + 3\vec{k}$ $[A/m^2]$, $H_y = H_z = 0$ *and* $\frac{\partial H_x}{\partial x} = 0$, *Find* H_x.

Ans. *since* $\nabla \times \vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\vec{i} + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\vec{j} + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\vec{k}$

and $\vec{J} = -10\vec{j} + 3\vec{k}$ $[A/m^2]$, $H_y = H_z = 0$, *then*

$$\nabla \times \vec{H} = \left(\frac{\partial H_x}{\partial z}\right)\vec{j} - \left(\frac{\partial H_x}{\partial y}\right)\vec{k} = -10\vec{j} + 3\vec{k}$$

$$\Rightarrow \left(\frac{\partial H_x}{\partial z}\right) = -10, \qquad \left(\frac{\partial H_x}{\partial y}\right) = -3$$

$$\left(\frac{\partial H_x}{\partial z}\right) = -10 \quad \Rightarrow \quad H_x = -10z + F(y)$$

$$\left(\frac{\partial H_x}{\partial y}\right) = \left(\frac{\partial F(y)}{\partial y}\right) = -3 \quad \Rightarrow \quad F(y) = -3y + C$$

$$\therefore \quad H_x = -3y - 10z + C ...\#\#\#$$