n-order PDE

Scope: Linear PDE with constant coefficient

$$\sum_{i=0,\,j=0}^{n,n} P_{ij}(x,y)\frac{\partial^{i+j}z}{\partial x^i \partial y^j} = f(x,y) \qquad : i+j \le n, \qquad \frac{\partial^0 z}{\partial x^0 \partial y^0} = z$$

$$Ex: \qquad ax\frac{\partial^2 z}{\partial x^2} + by\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = e^x ....Non-homogeneous$$

$$a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} = \sin(x)..........Homogeneous$$

$$[\qquad i+j=m \qquad for \quad all \quad PDE \quad terms.]$$

$$Laplace \quad equation: \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

$$Wave \quad equation: \quad \frac{\partial^2 z}{\partial x^2} - k^2\frac{\partial^2 z}{\partial t^2} = 0$$

$$Poisson \quad equation: \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = f(x,y)$$

Notation

$$D \overset{\Delta}{=} \frac{\partial}{\partial x}$$

$$D^{*} \overset{\Delta}{=} \frac{\partial}{\partial y}$$

$$\sum_{i=0,\,j=0}^{n,n} P_{ij}(x,y)\frac{\partial^{i+j} z}{\partial x^{i}\partial y^{j}} = f(x,y) \qquad : i+j \le n, \qquad \frac{\partial^{0} z}{\partial x^{0}\partial y^{0}} = z$$

$$F(D,D^{*})z = f(x,y)$$

$$Ex: \quad ax\frac{\partial^2 z}{\partial x^2} + by\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = e^{x} \quad \Leftrightarrow \quad axD^2 z + byDD^{*}z + Dz = e^{x}$$

$$a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} = \sin(x) \quad \Leftrightarrow \quad aD^2 z + bDD^{*}z + cD^{*2}z = \sin(x)$$

Complementary function for Constant Homogeneous Equation

$$a_0\left(\frac{\partial^n z}{\partial x^n}\right)+a_1\left(\frac{\partial^n z}{\partial x^{n-1}\partial y}\right)+\ldots+a_n\left(\frac{\partial^n z}{\partial y^n}\right) = 0$$

$$F(D,D^*)z = (a_0 D^n + a_1 D^{n-1}D^* + \ldots + a_n D^{*n})z = 0$$

$\textit{while} \quad a_i (i=0,1,\ldots n) \quad \textit{is} \quad \textit{constant}.$

Solution will has the following form.

$$z = \phi(y+mx)$$ $,\phi(y+mx): \quad \textit{arbitrary} \quad \textit{function}$

Since $\dfrac{\partial^n z}{\partial x^r \partial y^s} \overset{\Delta}{=} D^r D^{*s} z = m^r \phi^{(r+s)}(y+mx) \quad : \quad (r+s=n)$

then $[\phi^{(n)}(y+mx)](a_0 m^n + a_1 m^{n-1} + \ldots + a_n) = 0$

as $\phi(y+mx)$ *is arbitrary function, then*

$$(a_0 m^n + a_1 m^{n-1} + \ldots + a_n) = F(m,1) = 0$$

$\Rightarrow \quad \textit{solutions}: \quad m_1, m_2, \ldots m_n$

Case 1: multiplicity=1, $m_1 \neq m_2 \neq \dots \neq m_n$

Auxiliary function

$$(a_0 m^n + a_1 m^{n-1} + \dots + a_n) = F(m,1) = 0$$

$$\Rightarrow \quad solutions: \quad m_1 \neq m_2 \neq \dots \neq m_n$$

$$z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \dots + \phi_n(y + m_n x)$$

$Ex(14-1):$ *Find solution of*

$$\frac{\partial^2 z}{\partial x^2} + a\left(\frac{\partial^2 z}{\partial x \partial y}\right) - 6a^2\left(\frac{\partial^2 z}{\partial y^2}\right) = 0$$

$$Ans: \quad F(D, D^*)z = (D^2 + aDD^* - 6a^2 D^{*2})z = 0$$

$$\Rightarrow \quad F(m,1) = m^2 + am - 6a^2 = 0$$

$$\Rightarrow \quad m_1 = -3a, \quad m_2 = 2a$$

$$\therefore \quad z = \phi_1(y - 3ax) + \phi_2(y + 2ax) \dots \#\#\#$$

Case 2: multiplicity=r, $m_1=m_2=\ldots=m_r$

Auxiliary function

$$(a_0 m^n + a_1 m^{n-1} + \ldots + a_n) = F(m,1) = 0$$

$$\Rightarrow \quad solutions: \quad m_1 = m_2 = \ldots = m_r, \quad m_{r+1}, \ldots m_n$$

$$z = \phi_1(y + m_1 x) + x\phi_2(y + m_1 x) + \ldots + x^{r-1}\phi_r(y + m_1 x) + \phi_{r+1}(y + m_{r+1}x) + \ldots + \phi_n(y + m_n x)$$

$Ex(14-2)$: *Find solution of*

$$\frac{\partial^2 z}{\partial x^2} + 2a\left(\frac{\partial^2 z}{\partial x \partial y}\right) + a^2\left(\frac{\partial^2 z}{\partial y^2}\right) = 0$$

$$Ans: \quad F(D, D^*)z = (D^2 + 2aDD^* + a^2 D^{*2})z = 0$$

$$\Rightarrow \quad F(m,1) = m^2 + 2am + a^2 = 0$$

$$\Rightarrow \quad m_1 = m_2 = -a$$

$$\therefore \quad z = \phi_1(y - ax) + x\phi_2(y - ax) \ldots \#\#\#$$

Case 3: m_1, m_2: complex conjugate pair

Auxiliary function

$$(a_0 m^n + a_1 m^{n-1} + \ldots + a_n) = F(m,1) = 0$$

$$\Rightarrow \quad solutions: \quad m_1 = \alpha + j\beta, \quad m_2 = \alpha - j\beta$$

$$z_{1,2} = \phi(y + \alpha x + j\beta x) + \psi(y + \alpha x - j\beta x)$$

Generally $\phi(y + \alpha x + j\beta x) \; [\overset{\Delta}{=} \phi(+)], \; \psi(y + \alpha x - j\beta x) \; [\overset{\Delta}{=} \psi(-)]$ *are complex, then* $z_{1,2}$ *are also complex. To make* $z_{1,2}$ *to be real number, assume:* $[\phi(+) = \phi_r + j\phi_i]$ *is conjugate pair of* $[\psi(-) = \phi(-) = \phi_r - j\phi_i]$,

$$\{ \; \phi(+) + \phi(-) = 2\phi_r, \qquad j[\phi(+) - \phi(-)] = -2\phi_i \; \}$$

$$\therefore \quad z_{1,2} = [\phi_1(y + \alpha x + j\beta x) + \phi_1(y + \alpha x - j\beta x)] + j[\phi_2(y + \alpha x + j\beta x) - \phi_2(y + \alpha x - j\beta x)]$$

Ex(14 − 3): *Find solution for* $(D^4 - 6D^3D^* + 14D^2D^{*2} - 16DD^{*3} + 8D^{*4})z = 0$

Ans: $(m^4 - 6m^3 + 14m^2 - 16m + 8) = 0, \;\Rightarrow\; m_1 = m_2 = 2, m_3 = 1 + j, m_4 = 1 - j$

$$\therefore \quad z = \phi_1(y + 2x) + x\phi_2(y + 2x) + [\phi_3(y + 1 + j) + \phi_3(y + 1 - j)] + j[\phi_4(y + 1 + j) - \phi_4(y + 1 - j)] \ldots\#\#\#$$

Complementary function for non-homogeneous & constant &linear PDE

$$General \quad form \quad of \quad linear \quad PDE: \qquad F(D,D^*)z = 0 \qquad : constant \quad coefficient$$

$$2 \quad order \quad PDE: \qquad (AD^2 + 2HDD^* + BD^{*2} + 2FD + 2GD^* + C)z = 0$$

$$A, H, B, F, G, C : constants$$

$$Ex: \quad Conic \quad section: \qquad Ax^2 + 2Hxy + By^2 + 2Fx + 2Gy + Cz = 0$$

$$Elliptic \quad type: \qquad AB - H^2 > 0$$

$$Parabolic \quad type: \qquad AB - H^2 = 0$$

$$Hyperbolic \quad type: \qquad AB - H^2 < 0$$

$$Heat \quad conduction \quad eq.: \qquad \frac{\partial^2 z}{\partial x^2} - k\frac{\partial z}{\partial t} = 0$$

$$Telephone \quad eq.: \qquad \frac{\partial^2 e}{\partial x^2} - LC\frac{\partial^2 e}{\partial t^2} - (RC + GL)\frac{\partial e}{\partial t} - RCe = 0$$

Complementary function of PDE

PDE : $F(D,D^*)z=0$

Solution : $z=ce^{ax+by}$: a,b : to be determined constant, c: arbitrary constant

$$Dz \overset{\Delta}{=} \frac{\partial z}{\partial x} = \frac{\partial ce^{ax+by}}{\partial x} = a(ce^{ax+by}) = az$$

$$D^*z \overset{\Delta}{=} \frac{\partial z}{\partial y} = \frac{\partial ce^{ax+by}}{\partial y} = b(ce^{ax+by}) = bz$$

$\therefore \quad F(D,D^*)z=0 \quad \Rightarrow \quad F(a,b)ce^{ax+by}=0$

$\therefore \quad F(a,b)=0 \quad \Rightarrow \quad$ relation between (a,b)

$\therefore \quad z=\sum_{i=1}^{\infty} c_i e^{a_i x+b_i y} \quad : F(a_i,b_i)=0$

if $\quad F(D,D^*)=(D-\alpha_1 D^*-\beta_1)G(D,D^*)$

then $\quad F(a,b)=(a-\alpha_1 b-\beta_1)G(a,b)=0 \quad \Rightarrow \quad (a-\alpha_1 b-\beta_1)=0$

$\Rightarrow \quad a=\alpha_1 b+\beta_1 \quad \therefore \quad (a_i,b_i)=(\alpha_1 b_i+\beta_1,b_i)$

$$\Rightarrow \quad z_1=\sum_{i=1}^{\infty} c_i e^{a_i x+b_i y}=\sum_{i=1}^{\infty} c_i e^{(\alpha_1 b_i+\beta_1)x+b_i y}=e^{\beta_1 x}\sum_{i=1}^{\infty} c_i e^{b_i(\alpha_1 x+y)}=e^{\beta_1 x}\phi(\alpha_1 x+y)$$

$$z=\sum z_k$$

$Ex(14-4):$ *Find solution for*

$$F(D,D^*)z = (D+2D^*)(D-2D^*+1)(D^2-2DD^*-1)z = 0$$

$$Ans: \quad F(a,b) = (a+2b)(a-2b+1)(a^2-2ab-1) = 0 = (a_i - \alpha_1 b_i - \beta_1)G(a_i,b_i)$$

$$\Rightarrow \quad a+2b=0 \quad \Rightarrow \quad a=-b \quad \alpha_1 = -2, \ \ \beta_1 = 0$$

$$\Rightarrow \quad a-2b+1=0 \quad \Rightarrow \quad a = 2b-1, \quad \alpha_1 = 2, \ \ \beta_1 = -1$$

$$\Rightarrow \quad a^2-2ab-1=0 \quad \Rightarrow \quad a = b \pm \sqrt{b^2+1}$$

$$\because \quad z_1 = \sum_{i=1}^{\infty} c_i e^{(a_i x + b_i y)} = \sum_{i=1}^{\infty} c_i e^{(\alpha_1 b_i + \beta_1)x + b_i y} = e^{\beta_1 x}\sum_{i=1}^{\infty} c_i e^{b_i(\alpha_1 x + y)} = e^{\beta_1 x}\phi(\alpha_1 x + y)$$

$$\therefore \quad z = \sum z_k = \phi_1(-2x+y) + e^{-x}\phi_2(2x+y) + \sum_{i=1}^{\infty} c_i e^{(b_i + \sqrt{b_i^2+1})x + b_i y} + \sum_{i=1}^{\infty} d_i e^{(b_i - \sqrt{b_i^2+1})x + b_i y}$$

$For \quad multiplicity = i;$

$$F(D,D^*) = (D - \alpha_1 D^* - \beta_1)^i G(D,D^*)$$

$$\Rightarrow \quad z = e^{\beta_1 x}[\phi_1(\alpha_1 x + y) + x\phi_2(\alpha_1 x + y) + \ldots + x^{i-1}\phi_i(\alpha_1 x + y)]$$

$Ex(14-5): \quad Find \quad solution \quad for$

$$F(D,D^*)z = (D - D^* - 1)^3 z = 0$$

$$Ans: \quad F(a,b) = (a-b-1)^3 = 0 = (a_i - \alpha_1 b_i - \beta_1)G(a_i, b_i)$$

$$\Rightarrow \quad a - b - 1 = 0 \quad \Rightarrow \quad a = b + 1, \quad \alpha_1 = 1, \; \beta_1 = 1$$

$$\because \quad z_1 = \sum_{i=1}^{\infty} c_i e^{(a_i x + b_i y)} = \sum_{i=1}^{\infty} c_i e^{(\alpha_1 b_i + \beta_1)x + b_i y} = e^{\beta_1 x} \sum_{i=1}^{\infty} c_i e^{b_i(\alpha_1 x + y)} = e^{\beta_1 x}\phi(\alpha_1 x + y)$$

$$\therefore \quad z = \sum z_k = e^x[\phi_1(x+y) + x\phi_2(x+y) + x^2\phi_3(x+y)] \ldots \#\#\#$$

Particular Integral for PDE

$$For \quad constant \quad coefficient \qquad\qquad F(D,D^*) = f(x,y)$$

$$f(x,y) \quad : \quad Ae^{ax+by}, \quad A\sin(x,y), \quad A\cos(x,y),$$
$$Ae^{ax+by}\sin(x,y), \quad Ae^{ax+by}\cos(x,y)$$

$$z_p \quad could \quad be \quad find \quad using$$
$$"method \quad of \quad undetermined \quad coefficient".$$

$Ex(14-6):$ *Find solution for*

$$F(D,D^*)z=(D^2-2DD^*+D^{*2}-D+D^*)z=2e^{2x+3y}-\sin(x+2y)$$

$Ans:$ $F(a,b)=(a^2-2ab+b^2-a+b)=(a-b)(a-b-1)=0=(a_i-\alpha_1 b_i-\beta_1)G(a_i,b_i)$

$\Rightarrow \quad a-b=0 \quad \Rightarrow \quad a=b \quad \alpha_1=1,\ \beta_1=0$

$\Rightarrow \quad a-b-1=0 \quad \Rightarrow \quad a=b+1, \quad \alpha_1=1,\ \beta_1=-1$

$$\because \quad z_1=\sum_{i=1}^{\infty}c_i e^{(a_i x+b_i y)}=\sum_{i=1}^{\infty}c_i e^{(\alpha_1 b_i+\beta_1)x+b_i y}=e^{\beta_1 x}\sum_{i=1}^{\infty}c_i e^{b_i(\alpha_1 x+y)}=e^{\beta_1 x}\phi(\alpha_1 x+y)$$

Complementary function $z_c:$ $\quad z_c=\sum z_k=\phi_1(x+y)+e^{-x}\phi_2(x+y)$

Particular integral $z_p:$ $\quad z_{p1}=Ce^{2x+3y}, \quad z_{p2}=A\sin(x+2y)+B\cos(x+2y)$

$z_{p1}:$ $\quad Dz_{p1}=2z_{p1}, \quad D^*z_{p1}=3z_{p1}$

$$\Rightarrow \quad (D^2-2DD^*+D^{*2}-D+D^*)z_{p1}=(2^2-2*2*3+3^2-2+3)Ce^{2x+3y}=2e^{2x+3y}$$

$\Rightarrow \quad C=1 \quad \Rightarrow \quad z_{p1}=e^{2x+3y}$

$z_{p2}:$ $\quad Dz_{p2}=A\cos(x+2y)-B\sin(x+2y), \quad D^2z_{p2}=-A\sin(x+2y)-B\cos(x+2y)$

$\quad D^*z_{p2}=2A\cos(x+2y)-2B\sin(x+2y), \quad D^{*2}z_{p2}=-4A\sin(x+2y)-4B\cos(x+2y)$

$\quad DD^*z_{p2}=-2A\sin(x+2y)-2B\cos(x+2y)$

$$\Rightarrow \quad (D^2-2DD^*+D^{*2}-D+D^*)z_{p2}=-(A+B)\sin(x+2y)+(A-B)\cos(x+2y)=-\sin(x+2y)$$

$$\Rightarrow \quad A=B=\frac{1}{2} \quad \Rightarrow \quad z_{p2}=\frac{1}{2}\sin(x+2y)+\frac{1}{2}\cos(x+2y)$$

$$\therefore \quad z=z_c+z_p=\phi_1(x+y)+e^{-x}\phi_2(x+y)+e^{2x+3y}+\frac{1}{2}\sin(x+2y)+\frac{1}{2}\cos(x+2y)...\#\#\#$$

$Ex(14-7):$ *Find solution for*

$$F(D,D^*)z=(D^2-3DD^*+2D^{*2}-2D+3D^*+1)z=4e^x$$

Ans: $F(a,b)=(a^2-3ab+2b^2-2a+3b+1)=(a-2b-1)(a-b-1)=0=(a_i-\alpha_1 b_i-\beta_1)G(a_i,b_i)$

$\Rightarrow \quad a-2b-1=0 \quad \Rightarrow \quad a=2b+1 \quad \alpha_1=2,\ \beta_1=1$

$\Rightarrow \quad a-b-1=0 \quad \Rightarrow \quad a=b+1, \quad \alpha_1=1,\ \beta_1=1$

$$\because \quad z_1=\sum_{i=1}^{\infty}c_i e^{(a_i x+b_i y)}=\sum_{i=1}^{\infty}c_i e^{(\alpha_1 b_i+\beta_1)x+b_i y}=e^{\beta_1 x}\sum_{i=1}^{\infty}c_i e^{b_i(\alpha_1 x+y)}=e^{\beta_1 x}\phi(\alpha_1 x+y)$$

Complementary function $z_c:$ $\quad z_c=\sum z_k=e^x[\phi_1(2x+y)+\phi_2(x+y)]$

Particular integral $z_p:$ $\quad z_p=Ce^x \quad \Rightarrow \quad Dz_p=z_p, \quad D^*z_p=0$

$\Rightarrow \quad (D^2-3DD^*+2D^{*2}-2D+3D^*+1)z_p=(1^2-0+0-2+0+1)Ce^x=0$

$[\because \quad z_p=Ce^x \sim e^x[\phi_1(2x+y)+\phi_2(x+y)]=z_c,$ *when* ϕ_1,ϕ_2 *are constants.*]

$\therefore$ *assume* $z_p=Cxe^x \quad \Rightarrow \quad Dz_p=C(1+x)e^x,\ D^2z_p=C(2+x)e^x,\ D^*z_p=D^{*2}z_p=DD^*z_p0$

$\Rightarrow \quad (D^2-3DD^*+2D^{*2}-2D+3D^*+1)z_p=[(2+x)-0+0-2(1+x)+0+x]Ce^x=0$

$[\because \quad z_p=Cxe^x \sim e^x[\phi_1(2x+y)+\phi_2(x+y)]=z_c,$ *when* $\phi_1=K(2x+y),\phi_2=-K(x+y)]$

$\therefore$ *assume* $z_p=Cx^2e^x \quad \Rightarrow \quad Dz_p=Ce^x(x^2+2x),\ D^2z_p=Ce^x(x^2+4x+2),$

$D^*z_p=D^{*2}z_p=DD^*z_p0$

$\Rightarrow \quad (D^2-3DD^*+2D^{*2}-2D+3D^*+1)z_p=[(x^2+4x+2)-0+0-2(x^2+2x)+0+x^2]Ce^x=2Ce^x$

$\Rightarrow \quad \therefore \quad C=2, \quad z_p=2x^2e^x$

$\therefore \quad z=z_c+z_p=e^x[\phi_1(2x+y)+\phi_2(x+y)]+2x^2e^x$...###

$Ex(14-8):$ *Find solution for* $F(D,D^*)z=(D^2-3DD^*+2D^{*2}-2D+3D^*+1)z=4e^x$

Ans : $F(D,D^*)z=(D-D^*-1)(D-2D^*-1)z=4e^x$

assume $(D-2D^*-1)z=u \quad \Rightarrow \quad (D-D^*-1)u=4e^x \quad \Rightarrow \frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=u+4e^x$

Lagrange method : $\Rightarrow \quad \frac{dx}{1}=\frac{dy}{-1}=\frac{du}{u+4e^x}$

1). $\frac{dx}{1}=\frac{dy}{-1} \quad \Rightarrow \quad x+y=c_1$

2). $\frac{dx}{1}=\frac{du}{u+4e^x} \quad \Rightarrow \quad \frac{du}{dx}-u=4e^x \quad \Rightarrow \quad u=4xe^x+c_2e^x \qquad [c_2=\phi(c_1)]$

$[\ \frac{du_c}{dx}-u_c=0 \ \Rightarrow \ \ln u_c=x+C \ \Rightarrow \ u_c=e^xe^C; \qquad u_p=Axe^x \ \Rightarrow \ u_p=4xe^x\ ]$

$\because \quad (D-2D^*-1)z=u \quad \Rightarrow \quad \frac{\partial z}{\partial x}-2\frac{\partial z}{\partial y}=z+(4xe^x+c_2e^x)$

Lagrange method : $\Rightarrow \quad \frac{dx}{1}=\frac{dy}{-2}=\frac{dz}{z+4xe^x+c_2e^x}$

1). $\frac{dx}{1}=\frac{dy}{-2} \quad \Rightarrow \quad 2x+y=c_3$

2). $\frac{dx}{1}=\frac{dz}{z+4xe^x+c_2e^x} \quad \Rightarrow \quad \frac{\partial z}{\partial x}-z=4xe^x+c_2e^x \quad \Rightarrow \quad z=e^x(2x^2+c_2x+c_4) \qquad [c_4=\phi_1(c_3)]$

$\Rightarrow \quad z=e^x[2x^2+x\phi(c_1)+\phi_1(c_3)]=e^x[2x^2+x\phi(x+y)+\phi_1(2x+y)]$

since $x=(2x+y)-(x+y)=c_3-c_1=c_5 \quad \Rightarrow \quad x\phi(x+y)=c_5\phi(x+y)=\phi_2(x+y)$

$\therefore \qquad z=e^x[2x^2+\phi_2(x+y)+\phi_1(2x+y)]...\#\#\#$

$$Ex(14-9): \quad Find \quad solution \quad for \quad F(D,D^*)z = (D^2 - DD^* - 6D^{*2})z = x + y$$

$$Ans:(method \quad 1) \qquad F(D,D^*)z = (D-3D^*)(D+2D^*)z = x+y$$

$$assume \qquad (D+2D^*)z = u \qquad \Rightarrow \qquad (D-3D^*)u = x+y \qquad \Rightarrow \frac{\partial u}{\partial x} - 3\frac{\partial u}{\partial y} = x+y$$

$$Lagrange \quad method: \qquad \Rightarrow \qquad \frac{dx}{1} = \frac{dy}{-3} = \frac{du}{x+y}$$

$$1). \; \frac{dx}{1} = \frac{dy}{-3} \qquad \Rightarrow \quad 3x + y = c_1 \qquad \Leftrightarrow \qquad y = c_1 - 3x$$

$$2). \; \frac{dx}{1} = \frac{du}{x+y} \qquad \Rightarrow \quad du = (x+y)dx = (x + c_1 - 3x)dx = (c_1 - 2x)dx$$

$$\Rightarrow \quad u = c_1 x - x^2 + c_2 \qquad = 2x^2 + xy + c_2 \qquad [c_2 = \phi_1(c_1) = \phi_1(y+3x)]$$

$$\because \qquad (D+2D^*)z = u \qquad \Rightarrow \qquad \frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = 2x^2 + xy + c_2$$

$$Lagrange \quad method: \qquad \Rightarrow \qquad \frac{dx}{1} = \frac{dy}{2} = \frac{dz}{2x^2 + xy + c_2}$$

$$1). \; \frac{dx}{1} = \frac{dy}{2} \qquad \Rightarrow \quad 2x - y = c_3 \qquad \Leftrightarrow \qquad y = 2x - c_3$$

$$2). \; \frac{dx}{1} = \frac{dz}{2x^2 + xy + c_2} \quad \Rightarrow \quad dz = (2x^2 + xy + c_2)dx \quad \Rightarrow \quad dz = (2x^2 + x(2x - c_3) + c_2)$$

$$\Rightarrow \qquad dz = (4x^2 - c_3 x + c_2) \quad \Rightarrow \quad z = \frac{4}{3}x^3 - \frac{c_3}{2}x^2 + c_2 x + c_4 \qquad [c_4 = \phi_2(c_3) = \phi_2(2x-y)]$$

$$\therefore \qquad z = \frac{4}{3}x^3 - \frac{(2x-y)}{2}x^2 + \phi_1(y+3x)x + \phi_2(2x-y) = \frac{1}{3}x^3 + \frac{y}{2}x^2 + \phi_1(y+3x)x + \phi_2(2x-y) ...\#\#\#$$

$$Ex(14-9): \quad Find \quad solution \quad for \quad F(D,D^*)z = (D^2 - DD^* - 6D^{*2})z = x + y$$

$Ans: (method \quad 2)$

$Complementary \quad function \quad z_c:$

$$F(D,D^*)z_c = (D-3D^*)(D+2D^*)z_c = 0 \quad \Rightarrow \quad F(m,1) = (m-3)(m+2) = 0$$

$$m_1 = 3, \ m_2 = -2 \quad \Rightarrow \quad z_c = \varphi_1(y+m_1x) + \varphi_2(y+m_2x) = \varphi_1(y+3x) + \varphi_2(y-2x)$$

$Particular \quad Integral \quad z_p:$

$$(D^2 - DD^* - 6D^{*2})z_p = x + y \quad \Rightarrow \quad z_p = (D^2 - DD^* - 6D^{*2})^{-1}(x+y)$$

$$z_p = D^{-2}\left[1 - \left\{\frac{D^*}{D} + 6\left(\frac{D^*}{D}\right)^2\right\}\right]^{-1}(x+y) \quad = D^{-2}\left[1 + (\frac{D^*}{D} + 6(\frac{D^*}{D})^2) + \ldots\right](x+y)$$

$$[(1-x)^{-1} = (1 + x + x^2 + \ldots), \quad D^*(x+y) = 1, \quad \frac{1}{D}(1) = \int \partial x = x, \quad D^{*n}(x+y) = 0 \ \ n \geq 2]$$

$$z_p = D^{-2}[(x+y) + x + 0 + 0\ldots] \quad = D^{-1}[D^{-1}(2x+y)] \quad = \int \left(\int (2x+y)dx \right)\Big|_{y=const} dx$$

$$= \int (x^2 + xy)dx \quad = \frac{x^3}{3} + \frac{x^2 y}{2}$$

$$z = z_c + z_p \quad = \varphi_1(y+3x) + \varphi_2(y-2x) + \frac{x^3}{3} + \frac{x^2 y}{2} \ldots \#\#\#$$

$Ex(14-10).$ *Voltage(e) and current(i) in transmission line follows*

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad : u \;\; represents \;\; e, \; i \qquad c : wave \;\; velocity$$

$$if \;\; u(x,0) = f(x) \;\; and \;\; \left.\frac{\partial u(x,0)}{\partial t}\right|_{t=0} = g(x), \;\; find \;\; u(x,t)$$

$$Ans. \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \Rightarrow \quad F(D, D^*)u = (D^2 - \frac{D^{*2}}{c^2})u = 0 \qquad [D = \frac{\partial}{\partial x}, \qquad D^* = \frac{\partial}{\partial t}]$$

$$F(m,1) = m^2 - \frac{1}{c^2} = 0 \quad \Rightarrow \quad m_1 = \frac{1}{c}, \qquad m_2 = -\frac{1}{c}$$

$$\therefore \quad u(x,t) = \phi_1(t + m_1 x) + \phi_2(t + m_2 x) \quad = \phi_1(t + \frac{x}{c}) + \phi_2(t - \frac{x}{c}) \quad = \psi_1(x+ct) + \psi_2(x-ct)$$

$$\Rightarrow \quad \frac{\partial u(x,t)}{\partial t} = c\psi_1'(t + \frac{x}{c}) + \psi_2'(t - \frac{x}{c})$$

$$t = 0: \quad u(x,0) = \psi_1(x) + \psi_2(x) \quad = f(x)$$

$$\frac{\partial u(x,0)}{\partial t} = c\psi_1'(x) + \psi_2'(x) \quad = g(x) \quad \Rightarrow \int \Rightarrow \quad c\psi_1(x) + c\psi_2(x) \quad = \int_b^x g(v)dv$$

$$\Rightarrow \quad \psi_1(x) = \frac{1}{2}\left[f(x) + \frac{1}{c}\int_b^x g(v)dv\right]; \qquad \psi_2(x) = \frac{1}{2}\left[f(x) - \frac{1}{c}\int_b^x g(v)dv\right]$$

$$\therefore \quad u(x,t) = \psi_1(x+ct) + \psi_2(x-ct) \quad = \frac{1}{2}\left[f(x+ct) + f(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} g(v)dv\right] ...\#\#\#$$

$\Rightarrow$ *"d' Alembert's solution"*