Power Series Solution

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0$$

Solution :

$$y(x) = (x-a)^r [a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots]$$

$$= \sum_{m=0}^{\infty} a_m (x-a)^{m+r}$$

a : *known constant*

r, a_m : *unknown constant*

$m = 0,\ 1,\ 2,\ \ldots$

Taylor series : $\sum_{m=0}^{\infty} a_m (x-a)^{m+r}$ .... $r = 0$ *or integer*

Power Series Solution

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0$$

$x = a$ *is* "*ordinary point*" *of ODE,*

if both $P(x)$ *and* $Q(x)$ *are Taylor series.*

$$P(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + \ldots$$

$$Q(x) = B_0 + B_1(x-a) + B_2(x-a)^2 + \ldots$$

another point is called "*singular point*"

If $x = a$ *is singular point, but both*

$$(x-a)P(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + \ldots \quad : \textit{Taylor series}$$

$$(x-a)^2 Q(x) = B_0 + B_1(x-a) + B_2(x-a)^2 + \ldots \quad : \textit{Taylor aeries}$$

then $x = a$ *is a* "*regular singular point*" *of ODE.*

Power Series Solution

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0$$

$$If \quad [\ (P(x) = \pm\infty) \quad and/or \quad (Q(x) = \pm\infty)\ ]_{x=a}$$

$$then \quad [x = a \quad is \quad a \quad singular \quad point \quad of \quad the \quad ODE.]$$

Power Series Solution

$$Ex.(5-1) \qquad y'' + \left(\frac{2}{x}\right)y' + \left\{\frac{3}{x(x-1)^3}\right\}y = 0$$

$$P(x) = \frac{2}{x} = 2x^{-1}$$

$$Q(x) = \frac{3}{x(x-1)^3} = -3x^{-1}(1-x)^{-3} = -3x^{-1}(1+3x+6x^2+\ldots)$$

$P(x)$ and $Q(x)$ are not Taylor series

then $(x=0)$ is singular point of the ODE.

$$(x-0)P(x) = x\frac{2}{x} = 2$$

$$(x-0)^2 Q(x) = (x-0)^2 \frac{3}{x(x-1)^3} = -3x(1+3x+6x^2+\ldots)$$

Both $P(x)$ and $Q(x)$ are Taylor series

then $(x=0)$ is regular singular point of the ODE.

Power Series Solution

$$Ex.(5-1.cont.) \qquad y'' + (\frac{2}{x})y' + \left\{\frac{3}{x(x-1)^3}\right\}y = 0$$

$$P(x) = \frac{2}{x} = \frac{2}{1+(x-1)} = 2[1-(x-1)+(x-1)^2 - \ldots]$$

$$Q(x) = \frac{3}{(x-1)^3[1+(x-1)]} = 3(x-1)^{-3}[1-(x-1)+(x-1)^2 + \ldots)$$

$P(x)$ *is a Taylor series but* $Q(x)$ *is not Taylor series then* $(x=1)$ *is another singular point of the ODE.*

$$(x-1)P(x) = 2(x-1)[1-(x-1)+(x-1)^2 - \ldots]$$

$$(x-1)^2 Q(x) = 3(x-1)^{-1}[1-(x-1)+(x-1)^2 + \ldots)$$

$(x-1)P(x)$ *is a Taylor series but* $(x-1)^2 Q(x)$ *is not Taylor series then* $(x=1)$ *is and irregular singular point of the ODE.*

Power Series Solution

$Ex.\qquad y'' + (\frac{2}{x})y' + \left\{\frac{3}{x(x-1)^3}\right\}y = 0$

1. $P(x) = \frac{2}{x} \quad , x=0 \Rightarrow P(0) = \infty, \qquad Q(x) = \frac{3}{x(x-1)^3} \quad , x=0 \Rightarrow Q(0) = \infty$

then $(x=0)$ *is singular point of the ODE.*

$$\lim_{x\to 0}(x-0)P(x) = \lim_{x\to 0}(x)(\frac{2}{x}) = 2 \neq \infty$$

$$\lim_{x\to 0}(x-0)^2 Q(x) = \lim_{x\to 0}(x^2)(\frac{3}{x(x-1)^3}) = \lim_{x\to 0}\frac{3x}{(x-1)^3} = 0 \neq \infty$$

$\therefore \qquad x=0$ *is regular singular point.*

2. $P(x) = \frac{2}{x} \quad , x=1 \Rightarrow P(1) = 2 \neq \infty, \qquad Q(x) = \frac{3}{x(x-1)^3} \quad , x=1 \Rightarrow Q(0) = \infty$

then $(x=1)$ *is singular point of the ODE.*

$$\lim_{x\to 1}(x-1)P(x) = \lim_{x\to 1}(x-1)(\frac{2}{x}) = 0 \neq \infty$$

$$\lim_{x\to 1}(x-1)^2 Q(x) = \lim_{x\to 1}(x-1)^2(\frac{3}{x(x-1)^3}) = \lim_{x\to 1}\frac{3}{x(x-1)} = \infty$$

$\therefore \qquad x=1$ *is irregular singular point.*

ทฤษฎีบทที่ 1

ถ้า $x=a$ เป็น ordinary point ของ

$$y''+P(x)y'+Q(x)y=0$$

ผลเฉลยแบบอนุกรมในรูปของนิพจน์การกระจายรอบจุด $x=a$

จะมีอยู่จริง และจะเป็น Taylor series คือ

$$y=a_0+a_1(x-a)+a_2(x-a)^2+\dots = \sum_{m=0}^{\infty} a_m (x-a)^m$$

โดยที่ค่า $|x-a|$ สูงสุดที่จะยังทำให้อนุกรมนี้ลู่เข้า(convergent)อยู่

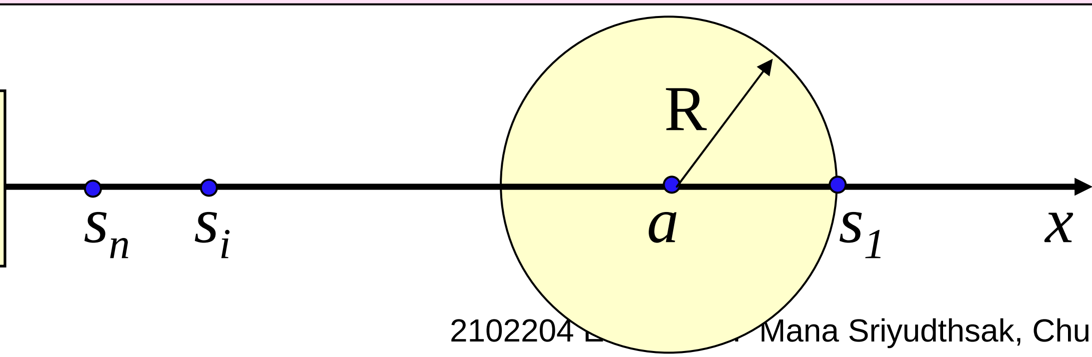
จะมีค่าไม่เกินระยะทางจากจุด $x=a$ ไปยัง singular point

ที่อยู่ใกล้กับจุด $x=a$ มากที่สุด

ระยะนี้เรียกว่า รัศมีของการลู่เข้า(radius of convergence: R)

a : ordinary point

s_n : singular point



ทฤษฎีบทที่ 2

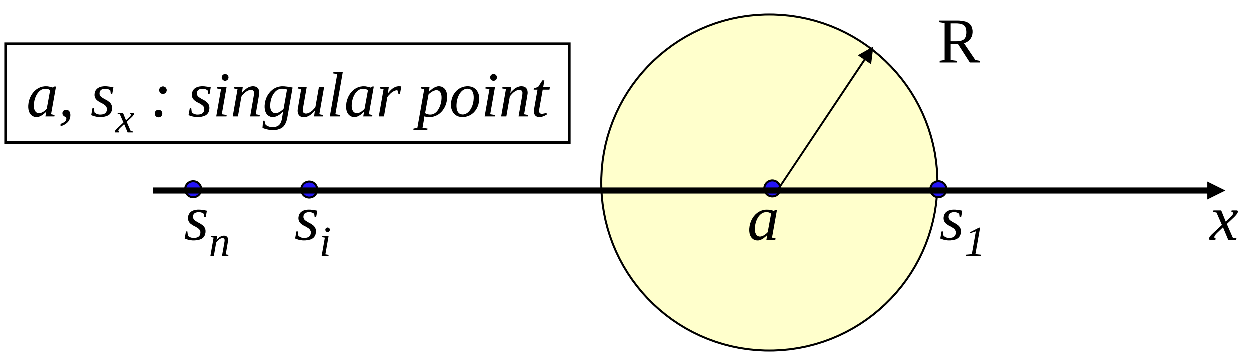
ถ้า $x=a$ เป็น *regular singular point* ของ

$$y'' + P(x)y' + Q(x)y = 0$$

ผลเฉลยแบบอนุกรมในรูปของนิพจน์การกระจายรอบจุด $x=a$ จะมีอยู่จริง และจะมีรูปแบบโดยทั่วไปเป็น

$$y = (x-a)^r[a_0 + a_1(x-a) + a_2(x-a)^2 + \dots\ ] = \sum_{m=0}^{\infty} a_m (x-a)^{m+r}$$

อนุกรมนี้จะลู่เข้าเมื่อ $0 < |x-a| < \mathrm{R}$ โดยที่ R เป็นระยะทางจากจุด $x=a$ ไปยัง *singular point* ที่อยู่ใกล้กับจุด $x=a$ มากที่สุด



ทฤษฎีบทที่ 3

ถ้า $x=a$ เป็น *irregular singular point* ของ

$$y''+P(x)y'+Q(x)y=0$$

โดยทั่วไปเราจะไม่สามารถหาผลเฉลยแบบอนุกรมของสมการนี้ออกมาได้ในรูปของนิพจน์การกระจายที่มีเฉพาะแต่พจน์ยกกำลังต่างๆ ของ $(x-a)$ *เท่านั้น*

นิพจน์ที่ได้จากการหาผลเฉลย มักจะเป็นอนุกรมที่ลู่ออก (divergent) ทุกค่า x *ยกเว้นที่* $x=a$

สรุป ทฤษฎีบทที่ 1-3

$$y'' + P(x)y' + Q(x)y = 0$$

$$if \quad (x=a) \quad is \quad ordinary \quad point \quad then$$

$$y(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + ... = \sum_{m=0}^{\infty} a_m (x-a)^m$$

$$if \quad (x=a) \quad is \quad regular \quad singular \quad point \quad then$$

$$y(x) = (x-a)^r[a_0 + a_1(x-a) + a_2(x-a)^2 + ...] = \sum_{m=0}^{\infty} a_m (x-a)^{m+r}$$

$$if \quad (x=a) \quad is \quad irregular \quad singular \quad point \quad then$$

$$y(x) \quad is \quad divergent \quad series.$$

Analytical Technique:

Find power series solution around

1. ordinary point at x=0

2. singular point at x=0

if ordinary point/singular point $x \neq 0$

then transform to x = 0.

Ex. $$y''(x) + \frac{2}{x} y'(x) + \frac{3}{x(x-1)^2} y(x) = 0$$

x=1 : singular point

assume z=x-1 or x=z+1 then

$$y''(z) + \frac{2}{z+1} y'(z) + \frac{3}{x^2(z+1)} y(z) = 0$$

Leibnitz-Maclaurin Method

$Ex.(5-2)\quad Find\ \ series\ \ solution\ \ of \quad (1-x^2)y'' - 5xy' - 3y = 0 \quad at\ \ ordinary\ \ point\ \ x = 0.$

$$Leibnitz's\ \ formula: \quad D^n(uv) = \sum_{m=0}^{n}\left[\frac{n!}{(n-m)!m!}\right]D^{n-m}(v)D^m$$

$$\Rightarrow \quad D^n[(1-x^2)y''] - 5D^n(xy') - 3D^n(y) = 0, \qquad assume: \quad u = (1-x^2), \quad v = y''$$

$$\therefore \quad D^0(u) = u = 1-x^2, \qquad D^1(u) = \frac{d}{dx}(1-x^2) = -2x, \qquad D^2(u) = D[D(u)] = \frac{d}{dx}(-2x) = -2,$$

$$D^3(u) = D^4(u) = ... = D^n(u)\big|_{n\geq 3} = 0$$

$$D(v) = D(y'') = y''', \quad D^2(v) = y^{(4)}, \quad .... \quad , \quad D^n(v) = y^{(n+2)}$$

$$\Rightarrow \quad D^n[(1-x^2)y''] = (1-x^2)y^{(n+2)} + n(-2x)y^{(n+1)} + \frac{n(n-1)}{2}(-2)y^{(n)}, \Rightarrow \quad D^n(xy') = xy^{(n+1)} + ny^{(n)}, \quad D^n(y) = y^{(n)}$$

$$\therefore \qquad (1-x^2)y^{(n+2)} - x(2n+5)y^{(n+1)} - (n+1)(n+3)y^{(n)} = 0$$

$$x = 0, \quad \Rightarrow \quad y^{(n+2)} - 0 \qquad - (n+1)(n+3)y^{(n)} = 0, \qquad \therefore \qquad y^{(n+2)} = (n+1)(n+3)y^{(n)}$$

$$n = 0, \Rightarrow \quad y'' = (1)(3)y(0), \qquad n = 1, \Rightarrow \quad y^{(3)} = (2)(4)y'(0),$$

$$n = 2, \Rightarrow \quad y^{(4)} = (3)(5)y''(0) = (1)(3)^2 5y(0) \qquad , n = 3, \Rightarrow \quad y^{(5)} = (4)(6)y^{(3)}(0) = 2(4)^2 6y'(0)$$

$$n = 4, \Rightarrow \quad y^{(6)} = (5)(7)y^{(4)}(0) = 1(3)^2(5)^2 7y(0) \qquad , n = 5, \Rightarrow \quad y^{(7)} = (6)(8)y^{(5)}(0) = 2(4)^2(6)^2 8y'(0)$$

$$Maclaurin\ \ series: \qquad y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + ...$$

$$\therefore \quad y(x) = \quad y(0)\left[1 + \frac{1(3)x^2}{2!} + \frac{1(3)^2 5x^4}{4!} + \frac{1(3)^2(5)^2 7x^6}{6!} + ...\right]$$

$$+ y'(0)\left[x + \frac{2(4)x^3}{3!} + \frac{2(4)^2 6x^5}{5!} + \frac{2(4)^2(6)^2 7x^7}{7!} + ...\right]...\#\#\#$$

Leibnitz-Maclaurin Method

$Ex.(5-2)$ *Find series solution of* $(1-x^2)y'' - 5xy' - 3y = 0$

at ordinary point $x = a.$

$$\therefore \quad (1-x^2)y^{(n+2)} - x(2n+5)y^{(n+1)} - (n+1)(n+3)y^{(n)} = 0$$

$x = a, \quad (1-a^2)y^{(n+2)}(a) - a(2n+5)y^{(n+1)}(a) - (n+1)(n+3)y^{(n)}(a) = 0$

$n = 0, \Rightarrow \quad y'' = [a(5)y' + (1)(3)y]/(1-a^2)$

$n = 1, \Rightarrow \quad y^{(3)} = [a(7)y'' + (2)(4)y']/(1-a^2) =$

$n = 2, \Rightarrow \quad y^{(4)} = [a(9)y^{(3)} + (3)(5)y^{(2)}]/(1-a^2)$

$n = 3, \Rightarrow \quad y^{(5)} = [a(11)y^{(4)} + (4)(6)y^{(3)}]/(1-a^2)$

$n = 4, \Rightarrow \quad y^{(6)} = [a(13)y^{(5)} + (5)(7)y^{(4)}]/(1-a^2)$

$n = 5, \Rightarrow \quad y^{(7)} = [a(15)y^{(5)} + (6)(8)y^{(5)}]/(1-a^2)$

$$y(x) = y(a) + (x-a)y'(a) + \frac{(x-a)^2}{2!}y''(a) + \frac{(x-a)^3}{3!}y'''(a) + \ldots..$$

$y(x) = \ldots\ldots\ldots\ldots\#\#$

Method of Frobenius *(Step 1)*

Step 1: Assume y(x) is a series solution around x=0 of

$$y'' + P(x)y' + Q(x)y = 0$$

then

$$y(x) = x^r(a_0 + a_1x + a_2x^2 + \ldots..) = \sum_{m=0}^{\infty} a_m x^{r+m}$$

hereby, x=0 is ordinary or regular singular point,

thus

$$xP(x) = b_0 + b_1x + b_2x^2 + \ldots.. \qquad \text{:Taylor series}$$

$$x^2Q(x) = c_0 + c_1x + c_2x^2 + \ldots.. \qquad \text{:Taylor series}$$

b_i, c_i : *known constant*

Method of Frobenius *(Step 2)*

Step 2: Find derivatives of y(x) [y'(x), y"(x)] in series form then replace in the ODE.

$$y(x)=x^r(a_0+a_1x+a_2x^2+\ldots..)$$

$$y'(x)=x^{r-1}[ra_0+(r+1)a_1x+(r+2)a_2x^2+\ldots..] = \sum_{m=0}^{\infty}(r+m)a_m x^{r+m+1}$$

$$y''(x)=x^{r-2}[r(r-1)a_0+(r+1)ra_1x+(r+2)(r+1)a_2x^2+\ldots..]$$

$$= \sum_{m=0}^{\infty}(r+m)(r+m1)a_m x^{r+m-2}$$

$$x^2 * ODE \quad \Rightarrow \qquad x^2y''+x[xP(x)]y'+x^2Q(x)y=0$$

$$[\quad xP(x)=b_0+b_1x+b_2x^2+\ldots.., \qquad x^2Q(x)=c_0+c_1x+c_2x^2+\ldots.. \quad]$$

$$\therefore \quad x^2\{x^{r-2}[r(r-1)a_0+(r+1)ra_1x+(r+2)(r+1)a_2x^2+\ldots..]\}$$

$$+x[b_0+b_1x+b_2x^2+\ldots..]\{x^{r-1}[ra_0+(r+1)a_1x+(r+2)a_2x^2+\ldots..]\}$$

$$+[c_0+c_1x+c_2x^2+\ldots..]\,x^r(a_0+a_1x+a_2x^2+\ldots..) = 0$$

Method of Frobenius *(Step 3)*

Step 3: Rearrange terms, using x^k as factor,

then find $r, a_0, a_1, a_2, \ldots.$

$$x^2 y'' + x[xP(x)]y' + x^2 Q(x) y = 0$$

$$\therefore\ a_0[r(r-1)+b_0 r+c_0]x^r + \{a_1[(r+1)r+b_0(r+1)+c_0]+a_0[b_1 r+c_1]\}x^{r+1}$$
$$+\{a_2[(r+2)(r+1)+b_0(r+2)+c_0]+a_1[b_1(r+1)+c_1]+a_0[b_2 r+c_2]\}x^{r+2}$$
$$+\ \ldots. \qquad = 0$$

Since $x \neq 0$ then coefficients must be 0.

$$a_0[r(r-1)+b_0 r+c_0] = 0$$

$$a_1[(r+1)r+b_0(r+1)+c_0]+a_0[b_1 r+c_1] = 0$$

$$a_2[(r+2)(r+1)+b_0(r+2)+c_0]+a_1[b_1(r+1)+c_1]+a_0[b_2 r+c_2] = 0$$

Method of Frobenius *Step 3 (cont.):*

Assume : $F(r) = r(r-1) + b_0 r + c_0$

i) if $(a_0 \neq 0)$: $a_0[r(r-1)+b_0 r+c_0] = 0 \Rightarrow a_0 F(r) = 0$

then $F(r) = [r(r-1)+b_0 r+c_0] = r^2+(b_0-1)r+c_0 = 0$

: “สมการดัชนี Indicial equation” of $y''+P(x)y'+Q(x)y = 0$

$\Rightarrow F(r) = (r-r_1)(r-r_2) = 0$ …… r_1, r_2 : “เลขชี้กำลัง(exponent)”

ii) if $(a_1 \neq 0)$: $a_1[(r+1)r+b_0(r+1)+c_0]+a_0[b_1 r+c_1] = 0$

$[F(r+1)=(r+1)r+ b_0(r+1) +c_0]$ $\therefore$ $a_1 = -a_0[b_1 r+c_1]/F(r+1)$

iii) if $(a_2 \neq 0)$:

$a_2[(r+2)(r+1)+b_0(r+2)+c_0]+a_1[b_1(r+1)+c_1]+a_0[b_2 r+c_2] = 0$

$[F(r+2)=(r+2)(r+1)+ b_0(r+2) +c_0]$

$\therefore a_2 = -\{a_0[b_2 r+c_2]+ a_1[b_1(r+1)+c_1]\}/F(r+2)$

$$\therefore a_m = -\{a_0[b_m r+c_m]+ a_1[b_{m-1}(r+1)+c_{m-1}]+\ldots + a_{m-1}[b_1(r+m-1)+c_1]\}/F(r+m)$$

Method of Frobenius *Step 3 (cont.):*

if $(r_1 \neq r_2)$ *and* $[\, F(r+1), F(r+2), \ldots, F(r+m)\,] \neq 0$

From $y(x) = x^r(a_0 + a_1 x + a_2 x^2 + \ldots..) = \sum_{m=0}^{\infty} a_m x^{r+m}$

then $y_1(x) = \sum_{m=0}^{\infty} a_m(r_1) x^{r_1+m} = x^{r_1} \sum_{m=0}^{\infty} a_m(r_1) x^m$

$$y_2(x) = \sum_{m=0}^{\infty} a_m(r_2) x^{r_2+m} = x^{r_2} \sum_{m=0}^{\infty} a_m(r_2) x^m$$

$$y(x) = c_1 y_2(x) + c_2 y_2(x)$$

$a_m(r)$: a_m *is a function of r*

if $(r_1 = r_2)$ *or* $(r_1 - r_2 = N$ *: integer*$)[\Rightarrow F(r_2+N) = F(r_1) = 0\,]$

Frobenius method will have only one solution….(have to find y_2*)*

Example : Method of Frobenius

Ex.(5-3): Find series solution around $x=0$ *for* $9x^2y''+(x+2)y=0$

Solution: $P(x)=0$, $Q(x)=(x+2)/9x^2$

$\therefore$ $Q(0)=\infty$, $x^2Q(x)=(x+2)$ $\Rightarrow$ $x=0$ *is regular singular point.*

then

$$y(x) = x^r(a_0+a_1x+a_2x^2+\ldots..)$$
$$y'(x) = x^{r-1}(ra_0+(r+1)a_1x+(r+2)a_2x^2+\ldots..)$$
$$y''(x) = x^{r-2}(r(r-1)a_0+(r+1)ra_1x+(r+2)(r+1)a_2x^2+\ldots..)$$

$$\Rightarrow \quad (a_0)\underline{[9r(r-1)+2]}x^r + \underline{\{a_1[9(r+1)r+2]+a_0\}}x^{r+1} + \underline{\{a_2[9(r+2)(r+1)+2]+a_1\}}x^{r+2} + \ldots = 0$$

$\therefore$ $\underline{9r(r-1)+2} = 0$, $\Rightarrow$ $9r^2-9r+2 = (r-1/3)(r-2/3) = 0$

$\therefore$ $\underline{a_1[9(r+1)r+2]+a_0} = 0$, $\Rightarrow$ $a_1 = -a_0/[9(r+1)r+2]$

$\therefore$ $\underline{a_2[9(r+2)(r+1)+2]+a_1} = 0$ $\Rightarrow$ $a_2 = -a_1/[9(r+2)(r+1)+2]$

$$\mathbf{a_{n+1} = -a_n/\{[3(r+n)+1)][3(r+n)+2]\}}$$

Example : Method of Frobenius (cont.)

Ex.(5-3): Find series solution around $x=0$ *for* $9x^2y''+(x+2)y=0$

Solution: at $r=r_1=1/3$ *assume* $a_0=a_{01}$ *then*

$a_1 = -a_0/[9(r+1)r+2] \quad = -a_{01}/[9(1/3+1)(1/3)+2] = -a_{01}/(2*3)$

$a_2 = -a_1/[9(r+2)(r+1)+2] = -a_1/[9(1/3+2)(1/3+1)+2]$

$\quad = -a_1/[5*6] \quad = a_{01}/(2*3*5*6)$

$a_3 = -a_2/[8*9] \quad = a_{01}/(2*3*5*6*8*9)$

$$y_1(x) = a_{01}x^{\frac{1}{3}}[1-\frac{x}{2*3}+\frac{x^2}{2*3*5*6}-\frac{x^3}{2*3*5*6*8*9}+....]$$

at $r=r_2=2/3$ *assume* $a_0=a_{02}$ *then*

$a_1 = -a_{01}/(3*4), \quad a_2 = -a_1/[6*7] = a_{01}/(3*4*6*7)$

$a_3 = -a_2/[9*10] = a_{01}/(3*4*6*7*9*10)$

$$y_2(x) = a_{02}x^{\frac{2}{3}}[1-\frac{x}{3*4}+\frac{x^2}{3*4*6*7}-\frac{x^3}{3*4*6*7*9*10}+....]$$

$$y(x) = y_1(x)+y_2(x)$$

Finding second solution

Finding second solution : $y = y_1 + y_2 = \phi * y_1$

$\therefore$

$$y_2(x) = y_1(x)\int \frac{e^{\int -P(x)dx}}{y_1^2(x)}dx$$

1. *If* $r_1 = r_2$ *then* $y_1 = y_2$

2. $r_1 - r_2 = N$ *N: Integer*

   Frobenius method will offer only one solution .

y_1 *at* $r = r_1$ *around regular singular point.*

[For series solution around ordinary point, Frobenius will offer 2 independent solutions.]

Case 1: $r_1=r_2$

$$Firstly \quad assume \quad r_1 \neq r_2, \qquad since \qquad y(x,r)=\sum_{m=0}^{\infty} a_m(r)x^{m+r}$$

$$r=r_1: \qquad y(x,r_1)=\sum_{m=0}^{\infty} a_m(r_1)x^{m+r_1}, \qquad r=r_2: \qquad y(x,r_2)=\sum_{m=0}^{\infty} a_m(r_2)x^{m+r_2}$$

$$assume: \qquad y_2(x)=\left[\frac{y(x,r_1)-y(x,r_2)}{r_1-r_2}\right] \qquad (r_1 \neq r_2)$$

$$r_2 \to r_1: \qquad y_2(x)=\lim_{r_2\to r_1}\left[\frac{y(x,r_1)-y(x,r_2)}{r_1-r_2}\right] = \left[\frac{\partial y(x,r)}{\partial r}\right]_{r=r_1}$$

$$=\left[\frac{\partial \sum_{m=0}^{\infty} a_m(r)x^{m+r}}{\partial r}\right]_{r=r_1} = \ell n|x|[\sum_{m=0}^{\infty} a_m(r_1)x^{m+r_1}] + \sum_{m=0}^{\infty} a_m^{'}(r_1)x^{m+r_1}]$$

$$[\frac{\partial x^r}{\partial r} = \ell n|x|(x^r) \qquad and \qquad \frac{\partial a_0}{\partial r}=0........]$$

$$y_2(x)=y_1(x)\ell n|x| \quad + \quad \sum_{m=0}^{\infty} K_m x^{m+r_1}]$$

Case 2: r_1-r_2=N

$$\textit{If } r_1 - r_2 = N \textit{ then } F(r_2+N) = F(r_1) = 0, \quad \Rightarrow \quad a_N(r_2) = \infty$$

(Frobenius slide17)

$$\textit{Assume} \quad Y(x,r) = (r-r_2)y(x,r) = (r-r_2)\sum_{m=0}^{\infty} a_m(r)x^{m+r}$$

$$Y(x,r_1) = (r_1-r_2)y(x,r_1) = Ny(x,r_1)........Y \quad depend \quad on \quad y$$

Follow Frobenius method then

$$(r-r_2)F(r) = (r-r_1)(r-r_2)^2 \qquad [r_2: \quad multiplicity = 2]$$

$$\textit{Then assume}: \quad y_2(x) = \left.\frac{\partial Y(x,r)}{\partial r}\right|_{r=r_2} = [\frac{\partial}{\partial r}\{(r-r2)y(x,r)\}]_{r=r2}$$

$$[\quad \lim_{r \to r_2}[(r-r_2)a_{N+n}(r)] = k_N a_N(r_1) \quad]$$

$$y_2(x,r_2) = k_N y_1(x)\ell n|x| + \sum_{m=0}^{\infty} K_m x^{m+r_2}$$

k_N, K_m *could be obtained from replace* y_2, y_2', y_2'' *in ODE......###*

$Ex.(5-4)\quad Find\quad series\quad solution\quad around\quad x=0\quad for$

$$x(x-1)y'' + (3x-1)y' + y = 0$$

$Solution:$

$Step\quad 1.\quad Check\quad P\quad and\quad Q$

$$P(x) = (3x-1)/[x(x-1)],$$

$$Q(x) = 1/[x(x-1)]$$

$$\Rightarrow P(0), Q(0) \to \infty, \quad xP(x), x^2 Q(x) : Taylor\quad series$$

$$\therefore \quad x = 0 \quad is\quad regular\quad singular\quad point.$$

$$\therefore \quad y(x) = \sum_{m=0}^{\infty} a_m x^{m+r},$$

$$y'(x) = \sum_{m=0}^{\infty} (r+m) a_m x^{m+r-1},$$

$$y''(x) = \sum_{m=0}^{\infty} (r+m)(r+m-1) a_m x^{m+r-2}$$

Step 2. Form the ODE in series.

Replace y, y', y'' in ODE:

$$\Rightarrow \sum_{m=0}^{\infty}(r+m)(r+m-1)a_m x^{m+r} - \sum_{m=0}^{\infty}(r+m)(r+m-1)a_m x^{m+r-1}$$

$$+3\sum_{m=0}^{\infty}(r+m)a_m x^{m+r} + \sum_{m=0}^{\infty}(r+m)a_m x^{m+r-1} + \sum_{m=0}^{\infty}a_m x^{m+r} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty}[(r+m)(r+m+2)+1]a_m x^{m+r} - r^2 a_0 x^{r-1}$$

$$-\sum_{m=1}^{\infty}(r+m)^2 a_m x^{m+r-1} = 0$$

$$Since \quad a_0 \neq 0 \quad and \quad all \quad coefficient \neq 0 \quad \Rightarrow \quad r^2 = 0,$$

$$r_1 = r_2 = 0$$

$$a_{m+1} = \frac{[(r+m)(r+m+2)+1]a_m}{(r+m+1)^2}$$

$$(r = r_1 = 0) \quad \Rightarrow \quad a_{m+1} = \frac{(m^2+2m+1)}{(m+1)^2} = a_m$$

$$\Rightarrow \quad a_0 = a_1 = a_2 = ...$$

$$y_1(x) = x^0(a_0 + a_1 x + a_2 x^2 + ...) = \frac{a_0}{(1-x)}.....$$

$$Since \quad r = r_1 = r_2$$

$$then \quad there \quad are \quad 2 \quad methods \quad for \quad determining \quad y_2(x).$$

Method 1: Using formular

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

$$\Rightarrow \qquad P(x) = \frac{(3x-1)}{x(x-1)} = \frac{1}{x} + \frac{2}{x-1}$$

$$\therefore \qquad \int P(x)dx = \int \left(\frac{1}{x} + \frac{2}{x-1} \right) dx \quad = \ln|x| + \ln(x-1)^2$$

$$= \ln\left| x(x-1)^2 \right|$$

$$\Rightarrow \qquad e^{-\int P(x)dx} = e^{-\ln\left| x(x-1)^2 \right|} \qquad = \frac{1}{x(x-1)^2}$$

$$\therefore \qquad y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx \qquad = \frac{a_0}{1-x} \int \frac{\frac{1}{x(x-1)^2}}{\left[\frac{a_0}{1-x} \right]^2} dx$$

$$= \frac{1}{a_o(1-x)} \int \frac{1}{x} dx \qquad = \frac{1}{a_0(1-x)} \ln|x| \ldots \# \# \#$$

$$Method \quad 2.1: \; Using \quad formular \quad (for \quad case \quad r_1 = r_2)$$

$$y_2 = \lim_{r_2 \to r_1} \frac{y(x,r_1) - y(x,r_2)}{r_1 - r_2} = \left[\frac{\partial y(x,r)}{\partial r}\right]_{r=r_1}$$

$$Since \quad y_1 = \frac{a_0}{1-x}$$

$$and \quad a_{m+1} = a_m = \ldots = a_0$$

$$then \quad y_2 = \left[\frac{\partial y(x,r)}{\partial r}\right]_{r=r_1} = \sum_{m=0}^{\infty} a_0 \left[\frac{\partial x^{m+r}}{\partial r}\right]_{r=r_1=0}$$

$$= \ln|x| \left(\sum_{m=0}^{\infty} a_0 x^m\right) = \ln|x| y_1(x)$$

$$= \frac{a_0}{1-x} \ln|x| \ldots \#\#\#$$

$Method\ \ 2.2:\ Using\ \ formular\ \ (for\ \ case\ \ \ r_1 = r_2)$

$$y_2 = y_1 \ln|x| + \sum_{m=1}^{\infty} K_m x^{r_1+m} \qquad then \qquad y_2' = y_1' \ln|x| + \frac{y_1}{x} + \sum_{m=1}^{\infty} (r_1+m) K_m x^{r_1+m-1}$$

$$y_2'' = y_1'' \ln|x| + 2\frac{y_1'}{x} - \frac{y_1}{x^2} + \sum_{m=1}^{\infty} (r_1+m)(r_1+m-1) K_m x^{r_1+m-2}$$

$Replace\ \ y_2,\ y_2',\ y_2''\ \ in\ \ ODE:$

$$\Rightarrow \qquad \ln|x|[x(x-1)y_1'' + (3x-1)y_1' + y_1] + 2(x-1)y_1' + 2y_1 + \sum(\ldots) + \sum(\ldots) + \sum(\ldots) = 0$$

$$\because \qquad [x(x-1)y_1'' + (3x-1)y_1' + y_1] = 0$$

$$and \qquad +2(x-1)y_1' + 2y_1 = -2\frac{a_0}{1-x} + 2\frac{a_0}{1-x} = 0$$

$$\Rightarrow \qquad \sum(\ldots) + \sum(\ldots) + \sum(\ldots)$$

$$= -(r_1+1)^2 K_1 x^{r1} + \sum_{m=1}^{\infty} \{[(r_1+m)(r_1+m+2)+1]K_m - (r_1+m+1)^2 K_{m+1}\} x^{r1+m} = 0$$

$replace\ \ r_1 = 0\ \ then\ \ set\ \ all\ \ coefficient\ \ to\ \ be\ \ 0.$

$$\Rightarrow \qquad K_1 = 0;\ \ K_m = K_{m+1} \qquad \Rightarrow \qquad K_1 = K_2 = K_3 = \ldots = 0$$

$$\therefore \qquad y_2(x) = y_1(x)\ln|x| + 0 \quad = \frac{a_0 \ln|x|}{1-x} \ldots\#\#\#$$

$Ex(5-5):\quad Find \quad series \quad solution \quad around \quad x=0 \quad for$

$$(x^2-1)x^2y'' - (x^2+1)xy' + (x^2+1)y = 0$$

$Solution: \quad Step1: \quad P(x) = \dfrac{-(x^2+1)x}{(x^2-1)x^2}, \quad x=0 \quad \rightarrow \quad P(x)=\infty$

$$Q(x) = \frac{-(x^2+1)}{(x^2-1)x^2}, \quad x=0 \quad \rightarrow \quad Q(x)=\infty$$

$$xP(x) = \frac{-(x^2+1)x}{(x^2-1)x} = 1+\ldots.. \qquad : Taylor \quad series$$

$$x^2Q(x) = \frac{-(x^2+1)}{(x^2-1)} = 1+\ldots.. \qquad : Taylor \quad series$$

$then \quad x=0 \quad is \quad regular \quad singular \quad point.$

$$\therefore \qquad y_2(x) = \sum_{m=0}^{\infty} a_m x^{r+m} \qquad and \quad using \quad Frobenius \quad method$$

$$F(r) = r^2 + (b_0-1)r + c_0 = r^2 - 1 = 0$$

$$\therefore \qquad r_1 = 1, \quad r_2 = -1 \qquad but \qquad r_1 - r_2 = 2 = integer$$

$***Can \quad not \quad use \quad conventional \quad technique***$

$$All\quad coefficient\quad relate\quad to\quad x\quad must\quad equal\quad 0,\quad then$$

$$[(r+1)^2-1]a_1=0 \quad\Rightarrow\quad a_1=0 \quad (r=r_1 \quad or \quad r_2)$$

$$a_{m+2}=\frac{a_m(r+m-1)^2}{(r+m+3)(r+m+1)} \quad\Rightarrow\quad a_1=a_3=a_5=\ldots a_{odd}=0$$

$$For\quad r=r_1=1, \qquad a_{m+2}=\frac{a_m(m)^2}{(m+4)(m+2)}$$

$$if\quad m=0\quad and\quad for\quad any\quad value\quad of\quad a_0, \quad a_2=0$$

$$then\quad a_2=a_4=a_6=\ldots=a_{even}=0$$

$$\therefore \quad y_1(x)=\sum_{m=0}^{\infty}a_m x^{r+m}=x^r\sum_{m=0}^{\infty}a_m x^m=x(a_0+0)=a_0x$$

$$For\quad r=r_2=-1, \qquad a_{m+2}=\frac{a_m(r+m-1)^2}{(r+m+3)(r+m+1)} \quad\Rightarrow\quad a_{m+2}=\frac{a_m(m-2)^2}{(m+2)(m)}$$

$$1.For\quad any\quad a_0\quad and\quad m=0,\ a_2=\infty\quad then\quad a_4=a_6=a_8=\ldots=\infty \Rightarrow Unsolved\quad y_2(x)$$

$$2.For\quad a\quad value\quad of\quad a_2\quad and\quad m=0, \Rightarrow a_0=0 \quad then\quad a_4=a_6=\ldots a_{even}=0$$

$$\Rightarrow \quad y_{r2}(x)=x^{r2}\sum_{m=0}^{\infty}a_m x^m=x^{-1}(a_2x^2)=a_2x......dependent\quad form\quad of\quad y_{r1}.$$

$Methed1: \quad Find \quad second \quad solution \quad using \quad formular \quad (for \quad case \quad r_1 - r_2 = N \quad : integer)$

$$y_2(x) = \left[\frac{\partial Y(x,r)}{\partial r}\right]_{r=r_2} = \left[\frac{\partial[(r-r_2)y(x,r)]}{\partial r}\right]_{r=r_2}$$

$Since \quad a_{m+2} = \dfrac{a_m(r+m-1)^2}{(r+m+3)(r+m+1)}$ ......$multiply \quad by \quad (r-r_2) = (r+1) \quad for \quad m = 0,2,4,...$

$$then \quad (r+1)a_2 = \frac{a_0(r-1)^2}{(r+3)}, \qquad (r+1)a_4 = \frac{a_2(r+1)^2}{(r+5)(r+3)} = \frac{a_0(r+1)^2(r-1)^2}{(r+5)(r+3)^2}$$

$$(r+1)a_6 = \frac{a_4(r+3)^2}{(r+7)(r+5)} = \frac{a_0(r+1)^2(r-1)^2}{(r+7)(r+5)^2}$$

$$\therefore \quad Y(x) = (r-r_2)y(x,r) = \sum_{m=0}^{\infty}(r+1)a_m x^{m+r}$$

$$= a_0 x^r\left[(r+1) + \frac{(r-1)^2x^2}{(r+3)} + (r+1)^2\left\{\frac{(r-1)^2x^4}{(r+5)(r+3)} + ...\right\}\right]$$

$$\therefore \quad y_2(x) = \left[\frac{\partial Y(x,r)}{\partial r}\right]_{r=r_2} = a_0 \ln|x|(x^{-1})[2x^2] + a_0(x^{-1})[1-3x^2]$$

$$= 2a_0[x\ln|x| + \frac{1}{2x}] - 3a_0x ...\#\#\#$$

$[term \quad 3a_0x \quad is \quad in \quad the \quad same \quad form \quad as \quad y_1 \quad then \quad could \quad be \quad neglected.]$