Analysis of laminates

Since the unidirectional lamina has poor transverse properties, it is hardly used in form of a single ply in the applications. Normally, several laminae are stacked together to form laminate. The fiber angle of each lamina can be designed such that the desired properties of the laminate are achieved.

If the fiber angles of plies 1, 2, 3 and 4 are $\theta_1, \theta_2, \theta_3, \text{ and } \theta_4$, respectively, the stacking sequence of this laminate can be written as $[\theta_1/\theta_2/\theta_3/\theta_4]$. For example, $[0/30/60/90]$ is a four-ply laminate with the fiber angle of $0^\circ$, $30^\circ$, $60^\circ$, and $90^\circ$, respectively. Additional rules are used to describe stacking sequence of laminate, as the following.

If two consecutive plies have the same fiber angle, subscripted number may be used.

$$[0/30/30/90] \equiv [0/30_s/90]$$

$\pm$ or $\mp$ can be used for consecutive plies

$$[45/-45/30/90] \equiv [\pm 45/30/90]$$

Subscribed “s” is used in case of symmetric laminate

$$[-30/60/60/-30] \equiv [-30/60]_s$$

$$[0/90/0/90/45/45/90/0/90/0] \equiv [(0/90)_2/45]_s$$

Overbar “−” indicates center ply for laminate with odd number of plies

$$\left[ (0/90)_2/45 \right]_s \equiv [0/90/0/90/45/90/0/90/0]$$
Theory of laminated beam in pure flexure

The following analysis is based on a paper by Pagano (1967)

Assumption:
1. Plane section remain plane
2. Symmetric stacking sequences (N.A. on midplane)
3. 0° or 90° fiber angle only (no shear couple)
4. Perfectly bonded plies ⇒ continuous strain

Strain at distance $z$ from N.A.,
$$
\varepsilon_x = \frac{(\rho + z)\phi - \rho\phi}{\rho\phi} = \frac{z}{\rho}
$$

where $\rho$ is a radius of curvature

It is noticed that the strain $\varepsilon_x$ is similar to isotropic beam. It is independent of material properties. However, stress $(\sigma_x = E_x\varepsilon_x)$ depends on material properties

At the $j^{th}$ ply,
$$
(\sigma_x)_j = (E_x)_j (\varepsilon_x)_j = (E_x)_j \frac{z}{\rho}
$$

Considering moment,

Force from stress $\sigma_x$ can be determined from $\sigma_x b dz$, where $b$ is the depth of beam.

Thus, moment can be determined from $(\sigma_x b dz)z$

The total moment
$$
M = 2\int_0^b \sigma_x dz
$$
\[ \frac{2b}{\rho} \int_0^b (E_x)_j z^2 \, dz \]
\[ \frac{2b}{\rho} \left( \frac{E_x}{3} \right)_j \left[ \frac{b}{2} \right] \]
\[ \frac{2b}{3\rho} \sum_{j=1}^N (E_x)_j \left( z_j^3 - z_{j-1}^3 \right) \]

For isotropic beam: \( \sigma = \frac{Mz}{I} \) or \( M = \frac{\sigma I}{z} \)

But \( \sigma = E_f \varepsilon = E_f \frac{z}{\rho} \)

Thus, \( M = \frac{E_f I}{\rho} = \frac{E_f h^3}{12\rho} \)

Compare (2) and (3) \( E_f = \frac{8}{h^3} \sum_{j=1}^N (E_x)_j \left( z_j^3 - z_{j-1}^3 \right) \)

Theory of laminated plates with coupling

Fundamental of strain-displacement relationship
Displacement of A is $u$ and $v$. So B displaces by $u + \frac{\partial u}{\partial x} dx$ and $v + \frac{\partial v}{\partial x} dx$ in the $x$ and $y$ direction, respectively. $u$ and $v$ are in terms of $x$ and $y$ or $u = u(x,y)$ and $v = v(x,y)$.

Thus, \[ \frac{A'B' - AB}{AB} = \varepsilon_x \quad \text{or} \quad A'B' = AB(\varepsilon_x + 1) \]

\[ (A'B')^2 = \left[d(x(\varepsilon_x + 1))\right]^2 = \left(dx + \frac{\partial u}{\partial x} dx\right)^2 + \left(dx + \frac{\partial v}{\partial x} dx\right)^2 \]

\[ dx^2 \left(\varepsilon_x^2 + 2\varepsilon_x + 1\right) = dx^2 \left(1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right) \]

Assume that product of displacement derivative $\to 0$

So, \[ \varepsilon_x = \frac{\partial u}{\partial x} \]

Similarly, \[ \varepsilon_y = \frac{\partial v}{\partial y} \quad \text{and} \quad \varepsilon_z = \frac{\partial w}{\partial z} \]

From figure above, \[ \tan \theta = \frac{\frac{\partial v}{\partial x} dx}{dx + \frac{\partial u}{\partial x} dx} \]

Since $\theta$ is very small, that is \[ \tan \theta = \theta \quad \text{or} \quad \theta = \frac{\frac{\partial v}{\partial x} dx}{dx + \frac{\partial u}{\partial x} dx} \]

\[ \frac{\partial u}{\partial x} \] is very small compared t unity, that is $dx \gg \frac{\partial u}{\partial x} dx$

That is \[ \theta = \frac{\partial v}{\partial x} \]

Similarly, \[ \lambda = \frac{\partial u}{\partial y} \]

So, share strain \[ \gamma_{xy} = \theta + \lambda = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \]
Similarly, 
\[ \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \]
and 
\[ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \]

**Classical Lamination Theory (CLT)**

**Objective:** To develop the stress-strain analysis for a multi-ply laminate. CLT is a combination of orthotropic elasticity, and classical thin plate theory.

**Thin Plate Theory.**

\[ t < \left\{ \frac{a}{10} \text{ and } \frac{b}{10} \right\} \]

Allowable plate loading

\[ N_x, N_y, N_{xy} \text{ are stress resultants with a unit of force/length. The stress resultant can be determined from } N_x = \sigma_{x,avg} t \]

Allowable moment loading
\( M_{xx}, M_{yy}, M_{xy}, M_{yx} \) are moment resultants with a unit of \( \text{force} \times \text{length} / \text{plate length} \) such as \( \text{N-m} / \text{m} \). \( M_{xx}, M_{yy} \) are bending moment and \( M_{xy}, M_{yx} \) are twisting moment.

Relating stress/moment resultants to plate stresses

From equilibrium of force, \( \sum F_x = 0 \)

We obtain \( N_x = \int_{-t/2}^{t/2} \sigma_x \, dz \)

From equilibrium of moment, \( \sum M_z = 0 \)

We obtain \( M_{xx} = \int_{-t/2}^{t/2} \sigma_z \, dz \)

Similarly: \( N_y = \int_{-t/2}^{t/2} \sigma_y \, dz \), \( N_{xy} = \int_{-t/2}^{t/2} \tau_{yx} \, dz = N_{yx} \)

\( M_{yy} = \int_{-t/2}^{t/2} \sigma_y \, dz \), \( M_{xy} = \int_{-t/2}^{t/2} \tau_{xy} \, dz = M_{yx} \)

Deformation of a thin plate (Kirchhoff Hypothesis)
Basic assumptions:

1. Plate is thin, i.e. $t << a, b$
2. Displacement $u, v$ and $w$ are small compared to thickness $t$
3. $\varepsilon_x, \varepsilon_y$, and $\gamma_{xy}$ are small compared to unity
4. $\gamma_{xz}$ and $\gamma_{yz}$ are negligible.
5. $u$ and $v$ are linear functions of $z$
6. $\varepsilon_z$ is negligible.
7. $\tau_{xz}$ and $\tau_{yz}$ vanish on the plate surfaces.

Kirchhoff Hypothesis: “A straight line initially perpendicular to mid plane of plate remains straight and perpendicular after deformation”

From figure above, $u_o$ is displacement in the x-direction of point $O$ or displacement of the mid plane.

Displacement of point $c$, $u_c = u_o - z_c \sin \alpha$

For small angle $\alpha$, $\sin \alpha = \tan \alpha = \alpha$

But $\tan \alpha = \frac{dw}{dx} \Rightarrow$ slope of mid plane

Thus, $u_c = u_o - z_c \frac{dw}{dx}$ or $u = u_o - z \frac{dw}{dx}$

Similarly, $v = v_o - z \frac{dw}{dy}$
From stress-strain relationship,

\[ \varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_o}{\partial x} - z \frac{d^2 w}{dx^2} = \varepsilon_x^o + z \kappa_x \]

where \( \kappa_x = -\frac{d^2 w}{dx^2} \) is the curvature of the middle surface.

Similarly,

\[ \varepsilon_y = \frac{\partial v}{\partial y} = \varepsilon_y^o + z \kappa_y \]

where \( \kappa_y = -\frac{d^2 w}{dy^2} \)

and

\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \]

\[ = \frac{\partial u_o}{\partial y} - z \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial v_o}{\partial x} - z \frac{\partial^2 w}{\partial x \partial y} \]

\[ = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y} \]

\[ = \gamma_{xy}^o + z \kappa_{xy} \]

Thus, the strain at any positions can be written in a matrix form as

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
= \begin{bmatrix}
\varepsilon_x^o \\
\varepsilon_y^o \\
\gamma_{xy}^o
\end{bmatrix} + z \begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]

This equation is important for determining strains at any points on the plate. If the mid-plane strains and curvatures are known, strains at any positions can be determined.

**Example:** A small segment of a four-layer laminate deformed such that a point \( P_o \) on the mid-plane has an extension strain in the x-direction of 1000 \( \mu \)strain. Radius of curvature \( R_o \) is 0.2 m. Determine \( \sigma_x \) through the thickness of the laminate if the laminate thickness is 0.6 mm.
Thus, the curvature $\kappa_x = \frac{1}{R_o} = \frac{1}{0.2} = 5 \, m^{-1}$

Strain $\varepsilon_x = \varepsilon_x^0 + z\kappa_x = 1000 + 5z$

At upper surface, $\varepsilon_x(z = -0.0003) = 1000 \times 10^{-6} + (-0.0003) \times 5$

$= -500 \, \mu\text{strain}$

At lower surface, $\varepsilon_x(z = +0.0003) = 1000 \times 10^{-6} + (0.0003) \times 5$

$= 2500 \, \mu\text{strain}$

Notice: Strain in laminates linearly varies with $z$ and independent of material properties

Laminate stress

From plate theory:

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0
\end{bmatrix} + z \begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]

Recall the ply stress-strain relationship
\[ \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [\boldsymbol{Q}] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \]

\{ \varepsilon \}_{xy} \text{ is function of } z \text{ and } [\boldsymbol{Q}] \text{ depends on material properties and fiber angle}

Thus, more appropriate stress-strain relationship is

\[ \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k = [\boldsymbol{Q}]_k \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}_k \]

where \( k \) refers to ply number

Thus, we obtain

\[ \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x^o + z\kappa_x \\ \varepsilon_y^o + z\kappa_y \\ \gamma_{xy}^o + z\kappa_{xy} \end{bmatrix}_k \]

This equation is not practical because curvature \( \kappa_x, \kappa_y, \text{ and } \kappa_{xy} \) are not generally known. So the mid-plane strains and curvatures should be related to the applied forces and bending moments using the equilibrium equation.

For example:

\[ N_z = \frac{t}{2} \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_z dz = \frac{t}{2} \int_{-\frac{t}{2}}^{\frac{t}{2}} \left[ \bar{Q}_{11} \varepsilon_x^o + \bar{Q}_{12} \varepsilon_y^o + \bar{Q}_{16} \gamma_{xy}^o + z\bar{Q}_{11} \kappa_x + z\bar{Q}_{12} \kappa_y + z\bar{Q}_{16} \kappa_{xy} \right] dz \]

\[ = \varepsilon_x^o \frac{t}{2} \int_{-\frac{t}{2}}^{\frac{t}{2}} \bar{Q}_{11} dz + \varepsilon_y^o \frac{t}{2} \int_{-\frac{t}{2}}^{\frac{t}{2}} \bar{Q}_{12} dz + \gamma_{xy}^o \frac{t}{2} \int_{-\frac{t}{2}}^{\frac{t}{2}} \bar{Q}_{16} dz + \ldots + \kappa_{xy} \frac{t}{2} \int_{-\frac{t}{2}}^{\frac{t}{2}} \bar{Q}_{16} \kappa_{xy} dz \]

\[ = \varepsilon_x^o \left\{ \bar{Q}_{11} \int_{z_0}^{z_1} \frac{dz}{z_{1}} + \bar{Q}_{11}^{(2)} \int_{z_1}^{z_2} \frac{dz}{z_{2}} + \bar{Q}_{11}^{(3)} \int_{z_2}^{z_3} \frac{dz}{z_{3}} + \ldots + \bar{Q}_{11}^{(n)} \int_{z_{n-1}}^{z_n} \frac{dz}{z_n} \right\} + \varepsilon_y^o \left\{ \bar{Q}_{12} \int_{z_0}^{z_1} \frac{dz}{z_{1}} + \bar{Q}_{12}^{(2)} \int_{z_1}^{z_2} \frac{dz}{z_{2}} + \bar{Q}_{12}^{(3)} \int_{z_2}^{z_3} \frac{dz}{z_{3}} + \ldots + \bar{Q}_{12}^{(n)} \int_{z_{n-1}}^{z_n} \frac{dz}{z_n} \right\} + \gamma_{xy}^o \left\{ \bar{Q}_{16} \int_{z_0}^{z_1} \frac{dz}{z_{1}} + \bar{Q}_{16}^{(2)} \int_{z_1}^{z_2} \frac{dz}{z_{2}} + \bar{Q}_{16}^{(3)} \int_{z_2}^{z_3} \frac{dz}{z_{3}} + \ldots + \bar{Q}_{16}^{(n)} \int_{z_{n-1}}^{z_n} \frac{dz}{z_n} \right\} \]
\[ + \kappa_x \left\{ \bar{O}_{11}^{(1)} \int_{z_0}^{z_1} zdz + \bar{O}_{11}^{(2)} \int_{z_1}^{z_2} zdz + \bar{O}_{11}^{(3)} \int_{z_2}^{z_3} zdz + \ldots + \bar{O}_{11}^{(n)} \int_{z_{n-1}}^{z_n} zdz \right\} \]

\[ + \kappa_y \left\{ \bar{O}_{12}^{(1)} \int_{z_0}^{z_1} zdz + \bar{O}_{12}^{(2)} \int_{z_1}^{z_2} zdz + \bar{O}_{12}^{(3)} \int_{z_2}^{z_3} zdz + \ldots + \bar{O}_{12}^{(n)} \int_{z_{n-1}}^{z_n} zdz \right\} \]

\[ + \kappa_{xy} \left\{ \bar{O}_{16}^{(1)} \int_{z_0}^{z_1} zdz + \bar{O}_{16}^{(2)} \int_{z_1}^{z_2} zdz + \bar{O}_{16}^{(3)} \int_{z_2}^{z_3} zdz + \ldots + \bar{O}_{16}^{(n)} \int_{z_{n-1}}^{z_n} zdz \right\} \]

Evaluate: \[ N_x = A_{11} \epsilon_x^o + A_{12} \epsilon_y^o + A_{16} \gamma_{xy}^o + B_{11} \kappa_x + B_{12} \kappa_y + B_{16} \kappa_{xy} \ldots \ldots \ldots \] **

where \[ A_{ij} = \sum_{m=1}^{n} \bar{O}_{ij}^{(m)} (z_m - z_{m-1}) \] and \[ B_{ij} = \frac{1}{2} \sum_{m=1}^{n} \bar{O}_{ij}^{(m)} (z_m^2 - z_{m-1}^2) \]

\( n \) is number of ply of the laminate.

The position of each ply \( z_i \) are graphically shown below

Consider the stress resultant in the \( y \)-direction and the shear stress resultant.

From \[ N_y = \int_{-t/2}^{t/2} \sigma_y dz \] and \[ N_{xy} = \int_{-t/2}^{t/2} \tau_{xy} dz = N_{yx} \]

We obtain \[ N_y = A_{12} \epsilon_y^o + A_{22} \epsilon_y^o + A_{26} \gamma_{xy}^o + B_{12} \kappa_x + B_{22} \kappa_y + B_{26} \kappa_{xy} \ldots \ldots \ldots \] **

\[ N_{xy} = A_{16} \epsilon_x^o + A_{26} \epsilon_y^o + A_{66} \gamma_{xy}^o + B_{16} \kappa_x + B_{26} \kappa_y + B_{66} \kappa_{xy} \ldots \ldots \ldots \] **

Similarly; \[ M_x = \int_{-t/2}^{t/2} \sigma_x zdz \]
\[ \begin{align*}
&= e_x^o \int_0^t \bar{Q}_{11} zdz + e_y^o \int_0^t \bar{Q}_{12} zdz + \gamma_{xy}^o \int_0^t z \bar{Q}_{16} zdz + \ldots + \kappa_{xy}^o \int_0^t \frac{z^2}{2} \bar{Q}_{16} zdz \\
&= B_{11} e_x^o + B_{12} e_y^o + B_{16} \gamma_{xy}^o + D_{11} \kappa_x + D_{12} \kappa_y + D_{16} \kappa_{xy} \ldots \ldots \ldots \ldots \ldots \ldots \***
\end{align*} \]

where \( D_{ij} = \frac{1}{3} \sum_{m=1}^3 \bar{Q}_{ij}^{(m)} (z_m^3 - z_{m-1}^3) \)

Write in matrix form

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\
B_{11} & B_{12} & B_{16} & B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} & B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66} & B_{16} & B_{26} & B_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x^o \\
\varepsilon_y^o \\
\gamma_{xy}^o \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
M \\
N
\end{bmatrix} =
\begin{bmatrix}
A & B \\
B & D
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x^o \\
\varepsilon_y^o
\end{bmatrix}
\]

where

\[
\begin{align*}
A & \equiv \text{Extensional stiffness, relate } \varepsilon \text{ to } N \\
B & \equiv \text{Coupling stiffness, relate } \kappa \text{ to } N, \text{ or } \varepsilon \text{ to } M \\
D & \equiv \text{Bending stiffness, relate } \kappa \text{ to } M
\end{align*}
\]

Analytical Approach:

1. Given material properties: \( E_1, E_2, G_{12}, v_{12}, \) and ply thickness
2. Stacking sequences.
3. Measured \( \varepsilon_x^o, \varepsilon_y^o, \gamma_{xy}^o, \kappa_x, \kappa_y, \) and \( \kappa_{xy} \)
4. Analyze
   a. Calculate \([Q]\) for each material
   b. Calculate \([\overline{Q}]\) for each ply
   c. Calculate \([ABD]\)
Coupling effects: Two types

1. Lamina shear coupling: Layer level

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]

These couplings are couplings of normal stresses to shear strain and shear stress to normal strain. These couplings lead to terms \( A_{16}, A_{26}, D_{16}, \) and \( D_{26} \) in \([ABD]\)

2. Laminate-level coupling: All \( B_{ij} \)

These couplings relate force resultants to curvatures and moment resultants to strains. For isotropic materials, \( A_{16}, A_{26}, D_{16}, \) and \( D_{26} \) equal zero, but \( B_{ij} \) may not be zero if their material properties are not symmetric with respect to the middle plane.

Stiffness characteristics of laminate

1. **Symmetric laminate**: Both geometric and material properties are symmetric about the middle surface

From \( B_{ij} = \frac{1}{2} \sum_{m=1}^{n} \bar{Q}_{ij}^{[m]} (z_m^2 - z_{m-1}^2) \)

For symmetric laminates, all \( B_{ij} = 0 \)

So, In-plane loads will not cause bending or twisting and bending or twisting moments will not cause mid-plane extension or contraction.

2. **Antisymmetric laminate**: Fiber angles of plies at the distance \( +z \) and \( -z \) are \( +\theta \) and \( -\theta \), respectively

- For example: [-45/45/-45/45] or \([\pm 45]_2\)
- $\theta$ can not be $0^\circ$ or $90^\circ$
- $A_{16} = A_{26} = D_{16} = D_{26} = 0$
- $B_{11} = B_{12} = B_{22} = B_{66} = 0$

3. **Quasi-isotropic laminate**: Laminates which exhibit some isotropic behavior.
   - The sub-matrix $[A]$ in $[ABD]$ is in the form of
     \[
     \begin{bmatrix}
     A_{11} & A_{12} & 0 \\
     A_{12} & A_{11} & 0 \\
     0 & 0 & \frac{A_{11} - A_{12}}{2}
     \end{bmatrix}
     \]
   - To obtain above property, it is required that
     1. Laminate is required to have three or more identical laminae
     2. Each lamina is $\pi/N$ apart where $N$ is number of plies

4. **Cross-ply laminate**: Laminates which plies orient at either $\theta = 0^\circ$ or $90^\circ$

5. **A balance cross-ply laminate**: Cross-ply laminate which has number of $90^\circ$ plies equal to number of $0^\circ$ plies.

6. **Angle-ply laminate**: Laminate which has ply angle as $+\theta$ or $-\theta$ only.

**Derivation and use of laminate compliances**

The \[
\begin{bmatrix}
M \\
N
\end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \varepsilon^o \\ \kappa \end{bmatrix}
\]
where $[ABD]$ may be called the laminate stiffness matrix is not practical since the applies loads or moments are usually known. It is more appropriate to write the equation in the form of
\[
\begin{bmatrix} \varepsilon^o \\ \kappa \end{bmatrix} = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix}
\]
where \[
\begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1}
\]
The \([abd]\) matrix can be approximate from the sub-matrix \([A]\), \([B]\), and \([D]\). However, with the advance in computation tool, one can perform the matrix inversion directly.

For a given applied loads and moments, mid-plane strains and curvature can be determined. Then, stresses in the laminate are determined from

\[
\{\sigma\}_k = [\overline{Q}]_k \{\{\varepsilon^o\} + z\{\kappa\}\}
\]

**Example** Determine the stiffness matrix \([ABD]\) for \([\pm 45]\) laminate consisting of 0.25-mm thick unidirectional A/S 3501 graphite/epoxy plies.

**Solution:**

It can be shown that

\[
[\overline{Q}]_{45} = \begin{bmatrix} 45.22 & 31.42 & 32.44 \\ 31.42 & 45.22 & 32.44 \\ 32.44 & 32.44 & 35.6 \end{bmatrix} \text{ GPa}
\]

and

\[
[\overline{Q}]_{-45} = \begin{bmatrix} 45.22 & 31.42 & -32.44 \\ 31.42 & 45.22 & -32.44 \\ 32.44 & 32.44 & 35.6 \end{bmatrix} \text{ GPa}
\]

Thus, \(A_{ij} = \sum_{m=1}^{n} [\overline{Q}^{(m)}_i] (z_m - z_{m-1}) = (\overline{Q}_j)_{45} t + (\overline{Q}_j)_{-45} t\) where \(t = 0.25 \text{ mm}\)

\(A_{11} = (45.22)(0.25) + (45.22)(0.25) = 22.61 \text{ GPa} \cdot \text{ mm} = A_{22}\)

\(A_{12} = (31.42)(0.25) + (31.42)(0.25) = 15.71 \text{ GPa} \cdot \text{ mm}\)

\(A_{16} = A_{26} = 0\)

\(A_{66} = (35.6)(0.25) + (35.6)(0.25) = 17.8 \text{ GPa} \cdot \text{ mm}\)
So, \[ A = \begin{bmatrix} 22.61 & 15.71 & 0 \\ 15.71 & 22.61 & 0 \\ 0 & 0 & 17.8 \end{bmatrix} \text{ GPa \cdot mm} \]

Similarly, \[ B_{ij} = \frac{1}{2} \sum_{m=1}^{n} \overline{Q}_{ij}^{(m)} \left( z_m^2 - z_{m-1}^2 \right) \]

\[ B_{11} = \frac{1}{2} \left[ (\overline{Q}_{11})_{45}(z_1^2 - z_0^2) + (\overline{Q}_{11})_{45}(z_2^2 - z_1^2) \right] = \frac{1}{2} \left[ 45.22 \left( 0^2 - (-0.25)^2 \right) + 45.22 \left( 0.25^2 - 0^2 \right) \right] = 0 = B_{22} \]

\[ B_{12} = \frac{1}{2} \left[ (\overline{Q}_{12})_{45}(z_1^2 - z_0^2) + (\overline{Q}_{12})_{45}(z_2^2 - z_1^2) \right] = 0 \]

\[ B_{16} = \frac{1}{2} \left[ 32.44 \left( 0^2 - (-0.25)^2 \right) + (-32.44) \left( 0.25^2 - 0^2 \right) \right] = -2.027 \text{ GPa \cdot mm}^2 \]

\[ B_{26} = B_{16} = -2.027 \text{ GPa \cdot mm}^2 \]

So, \[ B = \begin{bmatrix} 0 & 0 & -2.027 \\ 0 & 0 & -2.027 \\ -2.027 & -2.027 & 0 \end{bmatrix} \text{ GPa \cdot mm}^2 \]

Similarly, \[ D_{ij} = \frac{1}{2} \sum_{m=1}^{n} \overline{Q}_{ij}^{(m)} \left( z_m^3 - z_{m-1}^3 \right) \]

\[ D_{11} = \frac{1}{2} \left[ 45.22 \left( 0^3 - (-0.25)^3 \right) + 45.22 \left( 0.25^3 - 0^3 \right) \right] = 0.471 \text{ GPa \cdot mm}^3 \]

with similar calculation

\[ D = \begin{bmatrix} 0.471 & 0.327 & 0 \\ 0.327 & 0.471 & 0 \\ 0 & 0 & 0.371 \end{bmatrix} \text{ GPa \cdot mm}^2 \]

Example The \([\pm 45]\) laminate described in the previous example is subjected to a uniaxial force per unit length \(N_x = 30\) MPa-mm. Find the resulting stresses and strains in each ply along the \(x\) and \(y\) directions.
Solution:

From the previous example

\[
[ABD] = \begin{bmatrix}
22.61 & 15.71 & 0 & 0 & 0 & -2.027 \\
22.61 & 0 & 0 & 0 & -2.027 \\
17.8 & -2.027 & -2.027 & 0 \\
0.471 & 0.327 & 0 \\
Sym & 0.471 & 0 \\
& & & & 0.371
\end{bmatrix}
\]

\[
[abd] = [ABD]^{-1} = \begin{bmatrix}
0.1034 & -0.0415 & 0 & 0 & 0 & 0.3379 \\
0.1034 & 0 & 0 & 0 & 0 & 0.3379 \\
1.333 & 0.3386 & 0.3386 & 0 \\
4.959 & -1.986 & 0 \\
Sym & 4.959 & 0 \\
& & & & 6.387
\end{bmatrix}
\]

The mid-plane strains can be calculated as

\[
\begin{bmatrix}
\varepsilon_x^o \\
\varepsilon_y^o \\
\varepsilon_z^o \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix} = [abd] \begin{bmatrix}
30 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
0.003043 \text{ mm/mm} \\
-0.001183 \text{ mm/mm} \\
0 \\
0 \\
0 \\
0.01017 \text{ mm}^{-1}
\end{bmatrix}
\]

Therefore, stresses are calculated as follow
Strain distribution on the laminate (both plies)

\[
\begin{align*}
\varepsilon_x & = \varepsilon_x^o + z\kappa_x = 0.003043 \\
\varepsilon_y & = \varepsilon_y^o + z\kappa_y = -0.001183 \\
\gamma_{xy} & = \gamma_{xy}^o + z\kappa_{xy} = 0.01017z
\end{align*}
\]

Next, determine stress distribution on each ply

**Ply number 1 (+45° ply)**

\[
\begin{align*}
\{\sigma\}_x & = \left[\mathbf{Q}_{45^\circ}\right] \{\varepsilon\} = \\
\{\sigma\}_y & = \left[\mathbf{Q}_{45^\circ}\right] \{\varepsilon\} = \\
\tau_{xy} & = \left[\mathbf{Q}_{45^\circ}\right] \{\varepsilon\} = \\
& = \begin{bmatrix} 45.22 & 31.41 & 32.44 & 0.003043 \\ 45.22 & 32.44 & -0.001183 & 0.01017z \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
& = \begin{bmatrix} 0.1004 + 0.3299z \\ 0.0421 + 0.3297z \\ 0.0603 + 0.3621z \end{bmatrix} 
\end{align*}
\]

Thus, stresses at the bottom of ply number 1 (z = -0.25 mm) are

\[
\{\sigma\} = \begin{bmatrix} 0.01795 \\ -0.00403 \\ -0.03018 \end{bmatrix} \text{ GPa}
\]

and stresses at the top of ply number 1 (z = 0) are

\[
\{\sigma\} = \begin{bmatrix} 0.1004 \\ 0.0421 \\ 0.0603 \end{bmatrix} \text{ GPa}
\]

**Ply number 2 (-45° ply)**

\[
\begin{align*}
\{\sigma\}_x & = \left[\mathbf{Q}_{-45^\circ}\right] \{\varepsilon\} = \\
\{\sigma\}_y & = \left[\mathbf{Q}_{-45^\circ}\right] \{\varepsilon\} = \\
\tau_{xy} & = \left[\mathbf{Q}_{-45^\circ}\right] \{\varepsilon\} = \\
& = \begin{bmatrix} 0.10043 - 0.3299z \\ 0.04212 - 0.3297z \\ -0.0603 + 0.3621z \end{bmatrix}
\end{align*}
\]

Thus, stresses at the bottom of ply number 2 (z = -0.25 mm) are

\[
\{\sigma\} = \begin{bmatrix} 0.01795 \\ -0.00403 \\ -0.03018 \end{bmatrix} \text{ GPa}
\]

and stresses at the top of ply number 2 (z = 0) are

\[
\{\sigma\} = \begin{bmatrix} 0.1004 \\ 0.0421 \\ 0.0603 \end{bmatrix} \text{ GPa}
\]
Thus, stresses at the bottom of ply number 2 \((z = 0)\) are

\[ \{\sigma\} = \begin{bmatrix} 0.1004 \\ 0.00421 \\ -0.0603 \end{bmatrix} \text{ GPa} \]

and stresses at the top of ply number 1 \((z = 0.25 \text{ mm})\) are

\[ \{\sigma\} = \begin{bmatrix} 0.01795 \\ -0.04031 \\ 0.03017 \end{bmatrix} \text{ GPa} \]

In conclusion, stress distribution through the thickness of the laminate can be shown as

<table>
<thead>
<tr>
<th>Ply No.</th>
<th>z-coordinate</th>
<th>(\sigma_x) (MPa)</th>
<th>(\sigma_y) (MPa)</th>
<th>(\tau_{xy}) (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ((+45^\circ))</td>
<td>-0.25</td>
<td>17.95</td>
<td>-40.3</td>
<td>-30.18</td>
</tr>
<tr>
<td>2 ((-45^\circ))</td>
<td>0.25</td>
<td>17.95</td>
<td>-40.3</td>
<td>30.17</td>
</tr>
</tbody>
</table>

**Laminate Engineering Constant**

For a symmetric laminate subjected to in-plane loads, the stress resultants and strains are related according to

\[
\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{bmatrix}
\]

or

\[
\begin{bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{bmatrix} = \begin{bmatrix} A_{11}' & A_{12}' & A_{16}' \\ A_{12}' & A_{22}' & A_{26}' \\ A_{16}' & A_{26}' & A_{66}' \end{bmatrix} \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix}
\]

where \(A'_{ij}\) are members of the \([a bd]\)

Therefore, effective longitudinal Young modulus, \(E_x = \frac{\sigma_x}{\varepsilon_x^o}\) apply \(N_y\) only
where \( t \) is the thickness of the laminate

Similarly, \( E_y = \frac{\sigma_y}{\varepsilon_y} \) with \( N_y \) only

Effective laminate shear stress, \( G_{xy} = \frac{\tau_{xy}}{\gamma_{xy}} \) with \( N_{xy} \) only

Effective laminate Poisson’s ratio, \( v_{xy} = \frac{-\varepsilon_y}{\varepsilon_x} \) with \( N_x \) only

Similarly:

\[ \eta_{x,xy} = \frac{A'_{16}}{A_{11}} \quad \text{(}\gamma_{xy} \text{ due to } \sigma_x) \]

\[ \eta_{y,xy} = \frac{A'_{26}}{A_{66}} \quad \text{(}\varepsilon_y \text{ due to } \tau_{xy}) \]

**Hygrothermal Effects in laminate**

Assumptions:
1. No interaction between mechanical and hygrothermal effects
2. Uniform temperature and moisture distribution

Thin plate theory:

\[
\begin{bmatrix} \varepsilon_x' \\ \varepsilon_y' \\ \gamma_{xy}' \end{bmatrix} = \begin{bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{bmatrix} + z \begin{bmatrix} K_x \\ K_y \\ K_{xy} \end{bmatrix}
\]

where \( t \) refers to “total”

Strains with Hygrothermal effects may be written as

\[ \{ \varepsilon' \}_k = [\bar{S}]_k \{ \sigma \}_k + \{ \alpha \}_k \Delta T + \{ \beta \}_k C \]

or

\[ \{ \sigma \}_k = [\bar{Q}]_k \left( \{ \varepsilon' \}_k - \{ \alpha \}_k \Delta T - \{ \beta \}_k C \right) \]
\[
\begin{pmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{pmatrix}_k = \left[ \mathcal{Q} \right]_k
\begin{pmatrix}
\varepsilon'_x - \alpha_x \Delta T - \beta_x C \\
\varepsilon'_y - \alpha_y \Delta T - \beta_y C \\
\gamma_{xy} - \alpha_{xy} \Delta T - \beta_{xy} C
\end{pmatrix}
\]

With Hygrothermal effects; Resultant force \( N_x = \int_{-\frac{T}{2}}^{\frac{T}{2}} \sigma_x \, dz \)

\[
N_x = \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[ \mathcal{Q}_{11} \varepsilon_x^o + \mathcal{Q}_{12} \varepsilon_y^o + \mathcal{Q}_{16} \gamma_{xy}^o + z \mathcal{Q}_{11} \kappa_x + z \mathcal{Q}_{12} \kappa_y + z \mathcal{Q}_{16} \kappa_{xy} \right] \, dz
\]

\[
- \Delta T \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[ \mathcal{Q}_{11} \alpha_x + \mathcal{Q}_{12} \alpha_y + \mathcal{Q}_{16} \alpha_{xy} \right] \, dz - C \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[ \mathcal{Q}_{11} \beta_x + \mathcal{Q}_{12} \beta_y + \mathcal{Q}_{16} \beta_{xy} \right] \, dz
\]

Considering each integral

1\textsuperscript{st} Integral \( \equiv A_{11} \varepsilon_x^o + A_{12} \varepsilon_y^o + A_{16} \gamma_{xy}^o + B_{11} \kappa_x + B_{12} \kappa_y + B_{16} \kappa_{xy} \) (the same as the previous case)

2\textsuperscript{nd} Integral \( \equiv \Delta T \sum_{k=1}^{n} \left( z_k - z_{k-1} \right) \left[ \mathcal{Q}_{11} \alpha_x^{(k)} + \mathcal{Q}_{12} \alpha_y^{(k)} + \mathcal{Q}_{16} \alpha_{xy}^{(k)} \right] \) = \( N_x^T \). This integral is called the thermal stress resultant

3\textsuperscript{rd} Integral \( \equiv C \sum_{k=1}^{n} \left( z_k - z_{k-1} \right) \left[ \mathcal{Q}_{11} \beta_x^{(k)} + \mathcal{Q}_{12} \beta_y^{(k)} + \mathcal{Q}_{16} \beta_{xy}^{(k)} \right] \) = \( N_x^M \). This integral is called the hygroscopic stress resultant

Therefore,

\[
N_x = A_{11} \varepsilon_x^o + A_{12} \varepsilon_y^o + A_{16} \gamma_{xy}^o + B_{11} \kappa_x + B_{12} \kappa_y + B_{16} \kappa_{xy} - N_x^T - N_x^M
\]

Calculate other stress resultants, we obtain
Thermal stress and moment resultants \((N^T, M^T)\) are numerically equal to the mechanical stress moment resultant which would be required to induce strains exactly equal to the thermal strain.

**Laminate Failure Analysis**

Concept: 1. Failure of each ply occurs at a different load level.

2. After a ply fails, stiffness matrices [ABD] have to be modified.

3. Laminate can carry additional load even after many failures have occurred

Design Philosophies:

1. “First ply failure” design philosophy: Service loading does not cause any ply failure.

2. “Last ply failure” design philosophy: Service loading does not cause final ply failure.

Consider a stress resultant-strain curve of a laminate under uniaxial loading.
The total forces and moment is the summation of forces and moment from each section

\[
\begin{bmatrix}
N \\
M
\end{bmatrix}_{\text{Total}} = \sum_{n=1}^{k} \begin{bmatrix}
N^{(n)} \\
M^{(n)}
\end{bmatrix}
\]

Similarly, strains and curvature may be determined from

\[
\begin{bmatrix}
\varepsilon^o \\
\kappa
\end{bmatrix}_{\text{Total}} = \sum_{n=1}^{k} \begin{bmatrix}
\varepsilon^{o(n)} \\
\kappa^{(n)}
\end{bmatrix}
\]

The load deformation relationship for each section is

\[
\begin{bmatrix}
N^{(n)} \\
M^{(n)}
\end{bmatrix} = \begin{bmatrix}
A^{(n)} & B^{(n)} \\
B^{(n)} & D^{(n)}
\end{bmatrix}\begin{bmatrix}
\varepsilon^{o(n)} \\
\kappa^{(n)}
\end{bmatrix}
\]

where \([A^{(n)} B^{(n)} D^{(n)}]\) is the modifies stiffness matrices after the \((n-1)\)th ply failure

**Modified stiffness matrices:** 2 concepts

1) Some stiffnesses of the failed ply are remained

\[
\text{Failure}
\]

In this case of failure, \(E_2, G_{12},\) and \(v_{12} = 0\) but \(E_1 \neq 0\)

2) All stiffnesses are zero.

**Example**

A \([0/90/0]_s\) laminate consisting of AS/3501 graphite/epoxy laminae is subjected to uniaxial loading along the \(x\) direction. Use the maximum strain criterion to find the loads corresponding to first ply failure and ultimate failure, then plot the load-strain curve.

**Solution:**

Material properties of AS/3501 are: \(E_1 = 138\) GPa, \(E_2 = 9.0\) GPa, \(G_{12} = 6.9\) GPa, \(v_{12} = 0.3,\) thickness = 0.25 mm

\(S_L^{(r)} = 1448\) MPa, \(S_T^{(r)} = 48.3\) MPa
Thus, $\mathbf{Q} \mathbf{90} = \begin{bmatrix} 138.8 & 2.72 & 0 \\ 2.72 & 9.05 & 0 \\ 0 & 0 & 6.9 \end{bmatrix}$ and $\mathbf{Q} \mathbf{90}^\circ = \begin{bmatrix} 9.05 & 2.72 & 0 \\ 2.72 & 138.8 & 0 \\ 0 & 0 & 6.9 \end{bmatrix}$

$A_{ij}$ of $[0/90/0]_s$ is $A_{ij} = 4(0.25)\mathbf{Q} \mathbf{0} + 2(0.25)\mathbf{Q} \mathbf{90}^\circ$

$= [\mathbf{Q}] \mathbf{0} + 0.5[\mathbf{Q}] \mathbf{90}^\circ$

Without ply failures,

$$\begin{bmatrix} A_{ij}^{(1)} \end{bmatrix} = \begin{bmatrix} 143.3 & 4.08 & 0 \\ 4.08 & 78.45 & 0 \\ 0 & 0 & 10.35 \end{bmatrix} \text{ GPa-mm}$$

Failure strains are: $e_x^{(r)} = \frac{S_x^{(r)}}{E_1} = 0.0105$ and $e_y^{(r)} = \frac{S_y^{(r)}}{E_2} = 0.0054$

Therefore, first ply failure occurs at $90^\circ$ ply when $e_x^{(r)} = e_y^{(r)} = 0.0054$

The laminate force-deformation is

$$\begin{bmatrix} N_x^{(1)} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 143.3 & 4.08 & 0 \\ 4.08 & 78.45 & 0 \\ 0 & 0 & 10.35 \end{bmatrix} \begin{bmatrix} 0.0054 \\ e_y^{(1)} \\ \gamma_{xy}^{(1)} \end{bmatrix}$$

Solve,

$N_x^{(1)} = 0.773 \text{ GPa} \cdot \text{mm}$

$e_y^{(1)} = -0.000281$

$\gamma_{xy}^{(1)} = 0$

Modify $[A]$ to account for first ply failure; 2 approaches

1. All stiffnesses of $90^\circ$ ply are zero; $[\mathbf{Q}] \mathbf{90}^\circ = 0$

So, $[A^{(2)}] = [\mathbf{Q}] \mathbf{0} = \begin{bmatrix} 138.8 & 2.72 & 0 \\ 2.72 & 9.05 & 0 \\ 0 & 0 & 6.9 \end{bmatrix} \text{ GPa-mm}$

The second strain increment is

$e_x^{(2)} = e_x^{(r)} - e_x^{(1)} = 0.0105 - 0.0054 = 0.0051$
The incremental force-deformation

\[
\begin{bmatrix}
N_x^{(2)} \\
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
138.4 & 2.72 & 0 \\
2.72 & 9.05 & 0 \\
0 & 0 & 6.9
\end{bmatrix}
\begin{bmatrix}
0.0051 \\
\varepsilon_y^{(2)} \\
\gamma_{xy}^{(2)}
\end{bmatrix}
\]

Solve,

\[
N_x^{(2)} = 0.704 \text{ GPa} \cdot \text{mm}
\]
\[
\varepsilon_y^{(2)} = -0.00153
\]
\[
\gamma_{xy}^{(2)} = 0
\]

The ultimate failure load,

\[
N_x^{(Total)} = N_x^{(1)} + N_x^{(2)} = 0.773 + 0.704 = 1.477 \text{ GPa} \cdot \text{mm} \quad \text{Ans}
\]

2. Assume: \( E_2 = G_{12} = v_{12} = 0 \) but \( E_1 = 138 \text{ GPa} \) for the failed 90° ply

So, \( [Q]_{xy} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 138 & 0 \\ 0 & 0 & 0 \end{bmatrix} \)

and \( [A^{(2)}] = \begin{bmatrix} 138.8 & 2.72 & 0 \\ 2.72 & 78.05 & 0 \\ 0 & 0 & 6.9 \end{bmatrix} \) GPa-mm

So,

\[
\begin{bmatrix}
N_x^{(2)} \\
0 \\
0
\end{bmatrix} = [A^{(2)}] \begin{bmatrix}
0.0051 \\
\varepsilon_y^{(2)} \\
\gamma_{xy}^{(2)}
\end{bmatrix}
\]

Solve;

\[
N_x^{(2)} = 0.707 \text{ GPa} \cdot \text{mm}
\]
\[
\varepsilon_y^{(2)} = -0.000178
\]
\[
\gamma_{xy}^{(2)} = 0
\]

The ultimate failure load,

\[
N_x^{(Total)} = N_x^{(1)} + N_x^{(2)} = 0.773 + 0.707 = 1.48 \text{ GPa} \cdot \text{mm} \quad \text{Ans}
\]

The load-displacement curve is
Delamination due to interlaminar stress

The in-plane failure of laminated plate is introduced in the previous section. Another type of failure is failure in form of separation between plies which is called “delamination.” This mode of failure is caused by the so-called interlaminar stress. To derive the interlaminar stresses, consider the stress equilibrium.

Equilibrium equation: \( \sum F_x = 0 \)

\[
-\sigma_x dydz + \left[ \sigma_x + dx \frac{\partial \sigma_x}{\partial x} \right] dydz + \left( -\tau_{xy} \right) dxdz + \left[ \tau_{xy} + dy \frac{\partial \tau_{xy}}{\partial y} \right] dxdz \\
+ \left( -\tau_{xz} \right) dxdy + \left[ \tau_{xz} + dz \frac{\partial \tau_{xz}}{\partial z} \right] dxdy = 0
\]

Rearrange:

\[
\left( \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) dxdydz = 0
\]

or
\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \] .................................................................**

By considering the equilibrium in the other direction, we obtain
\[ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \] .................................................................**

and
\[ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \] .................................................................**

These are three equations called “stress equilibrium equations.” Consider a laminate plate under uniaxial loading,

At any point:
\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \]

\( \sigma_x \) is assumed to be constant along the \( x \)-direction. That is \( \frac{\partial \sigma_x}{\partial x} = 0 \).

So, to satisfy the equilibrium equation,
\[ \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \]

Or
\[ \tau_{xz}(z) = \int -\frac{\partial \tau_{xy}}{\partial y} \, dz \]

From CLT, \( \tau_{xy} \) is constant in the interior region. This leads to the conclusion that \( \tau_{xz} \) has to be zero according to the previous integral. However \( \tau_{xy} \) is vanished on the free edges. Thus there is a region where \( \tau_{xy} \) is decreased or \( \frac{\partial \tau_{xy}}{\partial y} \neq 0 \). Consequently, \( \tau_{xz} \neq 0 \) on the boundary region. This stress may cause delamination of the plates.