Static and Dynamic Characteristics of Signal
Input-Output Signal Concepts

Engineering tasks in the measurement of physical variables
(1) Selection of a measurement system
(2) Interpreting the output from a measurement system.

Selection of Measurement system based on input signal range

Static signal: range
Time-varying signal: magnitude and the rate of change of a variable
Signal Classification

- **Signal**: the physical information about a measured variable

### Continuous time, continuous value

Define for each instant of time and its amplitude may vary continuously with time and assume any value

- Analog signal

### Discrete time, continuous value

Define at discrete instants of time and its amplitude may vary continuously with time and assume any value

- Discrete time signal
Signal Classification

Discrete time, discrete value

Define at discrete instants of time and its amplitude may assume discrete value

• Digital signal
Signal Classification: waveform

• **Static signal**: does not vary in time (or change very slowly).

• **Dynamic signal**: is defined as a time-dependent signal

• **Deterministic signal**: varies in time in a predictable manner.
  - **Periodic signal**: the variation of magnitude of the signal repeats at regular intervals
  - **Non-Periodic signal**: does not repeat at regular interval.

• **Non-Deterministic signal**: cannot be prescribed before it occurs.
# Signal Classification: waveform

## Classification of Waveforms

<table>
<thead>
<tr>
<th>Type</th>
<th>Description</th>
<th>Formula</th>
</tr>
</thead>
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<tr>
<td>Static</td>
<td>y(t) = A&lt;sub&gt;0&lt;/sub&gt;</td>
<td></td>
</tr>
<tr>
<td>Dynamic</td>
<td></td>
<td></td>
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<tr>
<td>Periodic waveforms</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simple periodic</td>
<td></td>
<td>y(t) = A&lt;sub&gt;0&lt;/sub&gt; + C sin(ωt + φ)</td>
</tr>
<tr>
<td>Complex periodic</td>
<td></td>
<td>y(t) = A&lt;sub&gt;0&lt;/sub&gt; + ∑&lt;sub&gt;n=1&lt;/sub&gt; C&lt;sub&gt;n&lt;/sub&gt; sin(nωt + φ&lt;sub&gt;n&lt;/sub&gt;)</td>
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<tr>
<td>Aperiodic waveforms</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Step</td>
<td></td>
<td>y(t) = A&lt;sub&gt;0&lt;/sub&gt; U(t) = A&lt;sub&gt;0&lt;/sub&gt; for t &gt; 0</td>
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<tr>
<td>Ramp</td>
<td></td>
<td>y(t) = Kt for 0 &lt; t &lt; t&lt;sub&gt;f&lt;/sub&gt;</td>
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<tr>
<td>Pulse</td>
<td></td>
<td>y(t) = A&lt;sub&gt;0&lt;/sub&gt;U(t) - A&lt;sub&gt;0&lt;/sub&gt;U(t - t&lt;sub&gt;1&lt;/sub&gt;)</td>
</tr>
<tr>
<td>Nondeterministic</td>
<td></td>
<td></td>
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</table>
Periodic Signal

Mathematics of periodic signal

\[ x(t) = x(t + nT), \quad \text{for} \quad n = 1, 2, 3 \]

Where \( T \) is the period of the signal \( x(t) \)
Sinusoidal Signal

Generic sinusoidal signal: \[ x(t) = A \sin(\omega t + \phi) \]

Where \( A \) is the amplitude, \( \omega \) is the radian frequency and \( \phi \) is the phase.
**Signal Analysis**

- **Average or mean value during** $t_2 - t_1$
  
  **Analog**
  \[
  \bar{y} = \frac{\int_{t_1}^{t_2} y(t) \, dt}{\int_{t_1}^{t_2} dt}
  \]

- **root-mean-square value during** $t_2 - t_1$
  
  **Analog**
  \[
  y_{rms} = \sqrt{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} y(t)^2 \, dt}
  \]

- **Mean value**

  **Discrete or digital**
  \[
  \bar{y} = \frac{1}{N} \sum_{i=1}^{N} y_i
  \]

- **Amplitude of the ac component**

  **Discrete or digital**
  \[
  y_{rms} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} y_i^2}
  \]
RMS: Root-Mean-Square

RMS is a measure of a signal’s average power. Instantaneous power delivered to a resistor is $P = \frac{[v(t)]^2}{R}$.

$$P_{\text{avg}} = \frac{1}{R} \left( \frac{1}{T} \int_{t_0}^{t_0+T} [v(t)]^2 \, dt \right) = \frac{(V_{\text{rms}})^2}{R}$$

$$V_{\text{rms}} = \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} [v(t)]^2 \, dt}$$

An AC voltage with a given RMS value has the same heating (power) effect as an DC voltage with the same value.

All the following voltage waveforms have the same RMS value, and should indicate 1 V$_{\text{ac}}$ on an rms meter

<table>
<thead>
<tr>
<th>Waveform</th>
<th>Sine</th>
<th>Triangle</th>
<th>Square</th>
<th>DC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vpeak</td>
<td>1.414</td>
<td>1.733</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Vrms</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

All = 1 WATT
Signal Analysis

Ex Given a voltage signal as follows, the ac component has the frequency of 50 Hz, determine the average and rms values of signal that will be indicated by the meter when (a) S1 is open and (b) S1 is closed

Known signal characteristics

Find average and rms values
Signal are always associated with one or more systems. A system transforms the input into an output of the system.

\[
x(t) \quad \rightarrow \quad \text{System} \quad \rightarrow \quad y(t)
\]

Block diagram of a system

LTI is an mathematical operator

Linear, time invariant (LTI) system

A linear system is one which is both homogeneous (scale) and additive
A homogeneous system is one which a scaled input produces an equally scaled output.

An additive system is one for which

\[ mx(t) \rightarrow \text{System} \rightarrow my(t) \]

\[ x_1(t) + x_2(t) \rightarrow \text{System} \rightarrow y_1(t) + y_2(t) \]
A time invariant system is one for which a delay in the application of the input signal results in the same delay in the output.
A signal in frequency domain shows “How much” of a signal is associated with a certain frequency.

\[ x(t) = 2 \sin(2000 \pi t) \]

Time and frequency domain representation of the signal.
Comparing signals in the time and frequency domain

\[ x(t) = 1 + 2 \sin(600\pi t) + \cos(1000\pi t) \]
Fourier Series of Complex Periodic Signal

Fourier series for \( y(t) \):

\[
y(t) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos n\omega_0 t + B_n \sin n\omega_0 t \right)
\]

Where the fundamental frequency

\[
\omega_0 = \frac{2\pi}{T}
\]

\( A_0, A_n, B_n \) are constants that depend on \( y(t) \) and \( n \)

\[
A_0 = \frac{1}{T} \int_{-T/2}^{T/2} y(t) dt
\]

\[
A_n = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \cos n\omega_0 t dt; \quad n = 1, 2, 3, \ldots
\]

\[
B_n = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \sin n\omega_0 t dt; \quad n = 1, 2, 3, \ldots
\]
Useful Trigonometric Relations

\[
\begin{align*}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\sin A + \sin B &= 2 \sin \left( \frac{A + B}{2} \right) \cos \left( \frac{A - B}{2} \right) \\
\sin A - \sin B &= 2 \cos \left( \frac{A + B}{2} \right) \sin \left( \frac{A - B}{2} \right) \\
\cos A + \cos B &= 2 \cos \left( \frac{A + B}{2} \right) \cos \left( \frac{A - B}{2} \right) \\
\cos A - \cos B &= 2 \sin \left( \frac{A + B}{2} \right) \sin \left( \frac{B - A}{2} \right) \\
\sin A \sin B &= \frac{1}{2} \left[ \cos(A - B) - \cos(A + B) \right] \\
\cos A \cos B &= \frac{1}{2} \left[ \cos(A - B) + \cos(A + B) \right] \\
\sin A \cos B &= \frac{1}{2} \left[ \sin(A - B) + \sin(A + B) \right]
\end{align*}
\]
Useful Trigonometric Relations

Orthogonal principles

\[ \frac{1}{T} \int_0^T \sin \left( \frac{2\pi m}{T} \right) \sin \left( \frac{2\pi n}{T} \right) \, dt = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \\ 0 & m = n = 0 \end{cases} \]

\[ \frac{1}{T} \int_0^T \cos \left( \frac{2\pi m}{T} \right) \cos \left( \frac{2\pi n}{T} \right) \, dt = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \\ 1 & m = n = 0 \end{cases} \]

\[ \frac{1}{T} \int_0^T \sin \left( \frac{2\pi m}{T} \right) \cos \left( \frac{2\pi n}{T} \right) \, dt = 0 \]
Fourier Series of Periodic Signal

A function $f(t)$ is even if it is symmetric about the vertical axis

$$f(t) = f(-t)$$

A function $f(t)$ is odd if it is symmetric about the origin

$$f(t) = -f(-t)$$

Cosine is even function and Sine is odd function

Fourier Cosine Series

$$y(t) = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\omega_0 t)$$

Fourier Sine Series

$$y(t) = A_0 + \sum_{n=1}^{\infty} (B_n \sin n\omega_0 t)$$
Fourier Series of Periodic Signal

Ex Determine the Fourier series that represents the function as shown below

Known T = 10 and A_0 = 0 (Symmetrical)

Find Fourier coefficient A_n and B_n

Solution Since the function is odd, the Fourier series will contain only sine terms

\[ y(t) = \sum_{n=1}^{\infty} B_n \sin \frac{2n\pi t}{T} \]

\[ B_n = \frac{4}{n\pi}; \quad n = \text{odd number} \]

The resulting Fourier Series is then

\[ y(t) = 4 \sin \frac{2\pi}{10} t + 4 \sin \frac{6\pi}{10} t + 4 \sin \frac{10\pi}{10} t + \ldots \]
Fourier Series of Periodic Signal

\[ y(t) = \frac{4}{\pi} \sin \frac{2\pi}{10} t \]

\[ y(t) = \frac{4}{\pi} \sin \frac{2\pi}{10} t + \frac{4}{3\pi} \sin \frac{6\pi}{10} t + \frac{4}{5\pi} \sin \frac{10\pi}{10} t \]

\[ y(t) = \frac{4}{\pi} \sin \frac{2\pi}{10} t + \frac{4}{3\pi} \sin \frac{6\pi}{10} t + \frac{4}{5\pi} \sin \frac{10\pi}{10} t + \frac{4}{7\pi} \sin \frac{14\pi}{10} t \]
Fourier Series of Periodic Signal
Ex find the frequency content of the output voltage from a full-wave rectifier, if the ac input signal is given by

\[ E(t) = 120 \sin 120\pi t \]

**Known** The full-wave signal can be expressed as

\[ y(t) = |120 \sin 120\pi t| \]

**Find** The frequency content of this signal

**Solution** Since the function is even, the Fourier series will contain only cos terms
Fourier Series of Periodic Signal

The resulting Fourier Series is then

\[ y(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{2n\pi t}{T} \]

\[ A_0 = \frac{2 \times 120}{\pi} \]

\[ A_n = \frac{120}{\pi} \left( \frac{-2}{n-1} + \frac{2}{n+1} \right); \quad n = \text{even number} \]

\[ y(t) = \frac{2 \times 120}{\pi} + \frac{120}{\pi} \sum_{n=1}^{\infty} \frac{4}{1 - n^2} \cos 2\pi n 60t \]

\[ n = \text{even number} \]
\[
f(t) = \begin{cases} 
1 & 0 < t < \frac{T}{2} \\
-1 & -\frac{T}{2} < t < 0 
\end{cases}
\]

\[
f(t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \left( \frac{1}{n} \sin \frac{2n\pi}{T} t \right) \quad n = \text{odd number}
\]

\[
f(t) = \left| \frac{2}{T} t \right| = \begin{cases} 
\frac{2}{T} t & 0 < t < \frac{T}{2} \\
-\frac{2}{T} t & -\frac{T}{2} < t < 0 
\end{cases}
\]

\[
f(t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left( \frac{1}{n^2} \cos \frac{2n\pi}{T} t \right) \quad n = \text{odd number}
\]

\[
f(t) = \left| \sin \frac{2\pi}{T} t \right| = \begin{cases} 
\sin \frac{2\pi}{T} t & 0 < t < \frac{T}{2} \\
-\sin \frac{2\pi}{T} t & -\frac{T}{2} < t < 0 
\end{cases}
\]

\[
f(t) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=2}^{\infty} \left( \frac{1}{1-n^2} \cos \frac{2n\pi}{T} t \right) \quad n = \text{even number}
\]
\[ f(t) = \begin{cases} \sin \frac{2\pi}{T}t & 0 < t < \frac{T}{2} \\ 0 & -\frac{T}{2} < t < 0 \end{cases} \]

\[ f(t) = \frac{1}{\pi} + \frac{1}{2} \sin \frac{2\pi}{T}t + \frac{2}{\pi} \sum_{n=2}^{\infty} \left( \frac{1}{1-n^2} \cos \frac{2n\pi}{T}t \right) \quad n = \text{even number} \]

\[ f(t) = \begin{cases} 1 & 0 < t < \frac{T}{2} \\ 0 & -\frac{T}{2} < t < 0 \end{cases} \]

\[ f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left( \frac{1}{n} \sin \frac{2n\pi}{T}t \right) \quad n = \text{odd number} \]

\[ f(t) = \begin{cases} 1 & -\frac{T}{4} < t < \frac{T}{4} \\ 0 & -\frac{T}{2} < t < -\frac{T}{4} \quad \text{and} \quad \frac{T}{4} < t < \frac{T}{2} \end{cases} \]

\[ f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left( \sin \frac{n\pi}{n} \cos \frac{2n\pi}{T}t \right) \quad n = \text{odd number} \]
The Fourier transform is a mathematical function that transforms a signal from the time domain, \( y(t) \) to the frequency domain, \( Y(f) \).

**Definition of Fourier Transform of \( y(t) \)**

\[
Y(f) = \int_{-\infty}^{\infty} y(t) e^{-i2\pi ft} dt
\]

**Definition of Inverse Fourier Transform of \( Y(f) \)**

\[
y(t) = \int_{-\infty}^{\infty} Y(f) e^{i2\pi ft} df
\]
**Fourier Transform**

**Ex** Determine the Fourier transform, consider the one-sided exponential signal.

\[ x(t) = e^{-7t}u(t) \]

**Known** \( x(t) = e^{-7t}u(t) \)

**Find** Fourier Transform of \( x(t) \)

**Solution**

\[ X(f) = \mathcal{F}[x(t)] = \int_{0}^{\infty} e^{-7t} e^{-j2\pi ft} dt \]

\[ \mathcal{F}[u(t)e^{-7t}] = \frac{1}{7 + j2\pi f} \]
Impulse Function: Review

The function may be considered as a rectangular pulse of width $\varepsilon$ and height $1/\varepsilon$. In the limit $\varepsilon \to 0$, the height increase in such a way that the area is 1.

This leads to the definition

$$\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$$

Unit impulse function or Dirac delta function
Impulse Function: Review

\[ \delta(t-t_0) = \begin{cases} \infty; & t = t_0 \\ 0; & \text{elsewhere} \end{cases} \]

Shifting of function by \( t_0 \)

\[ \begin{align*}
\delta(t) &= \int_{-\infty}^{+\infty} e^{j2\pi ft} \, df \\
\delta(t-t_0) &= \int_{-\infty}^{+\infty} e^{j2\pi f(t-t_0)} \, df \\
\int_{-\infty}^{+\infty} f(t) \delta(t-t_0) \, dt &= f(t_0) \\
\delta(f) &= \int_{-\infty}^{+\infty} e^{-j2\pi ft} \, dt \\
\delta(f-f_0) &= \int_{-\infty}^{+\infty} e^{-j2\pi (f-f_0)t} \, dt \\
\int_{-\infty}^{+\infty} \delta(t) e^{-j2\pi ft} \, dt &= 1
\end{align*} \]
Discrete Fourier Transform

Discrete Fourier Transform (DFT) of $y_r$

$$y_r = y(r\delta t) \quad r = 1, 2, ..., N$$

$$Y(f_k) = \frac{2}{N} \sum_{r=1}^{N} y(r\delta t) e^{-i 2\pi k r / N} \quad k = 1, 2, ..., \frac{N}{2}$$

where $f_k = k\delta f$ and $\delta f = \frac{1}{N\delta t} = \frac{f_s}{N}$
Example: Estimate the amplitude spectrum or frequency content of the discrete data taken from \( y(t) = 10 \sin 2\pi t \) using a time increment of 0.125 s for the duration of 1 s.

Known: \( \delta t = 0.125 \text{ s} \) or \( f_s = 8 \text{ Hz} \)

Solution:
Discrete Fourier Transform

Discrete Data Set for \( y(t) = 10 \sin 2\pi t \)

<table>
<thead>
<tr>
<th>( r )</th>
<th>( y(r \delta t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.071</td>
</tr>
<tr>
<td>2</td>
<td>10.000</td>
</tr>
<tr>
<td>3</td>
<td>7.071</td>
</tr>
<tr>
<td>4</td>
<td>0.000</td>
</tr>
<tr>
<td>5</td>
<td>-7.071</td>
</tr>
<tr>
<td>6</td>
<td>-10.000</td>
</tr>
<tr>
<td>7</td>
<td>-7.071</td>
</tr>
<tr>
<td>8</td>
<td>0.000</td>
</tr>
</tbody>
</table>

\[ \delta f = \frac{f_s}{N} = \frac{8}{8} = 1 \text{ Hz} \]

\[ f_k = k \delta f = k \text{ Hz} \]

Discrete Fourier Transform of \( y(t) \)

| \( k \) | \( f_k \) (Hz) | \( Y(f_k) \) | \( |Y(f_k)| \) |
|--------|---------------|------------|-------------|
| 1      | 1             | -10i       | 10          |
| 2      | 2             | 0          | 0           |
| 3      | 3             | 0          | 0           |
| 4      | 4             | 0          | 0           |

Amplitude vs. Frequency

- Amplitude vs. Frequency
- Frequency Hz
- Amplitude |V|