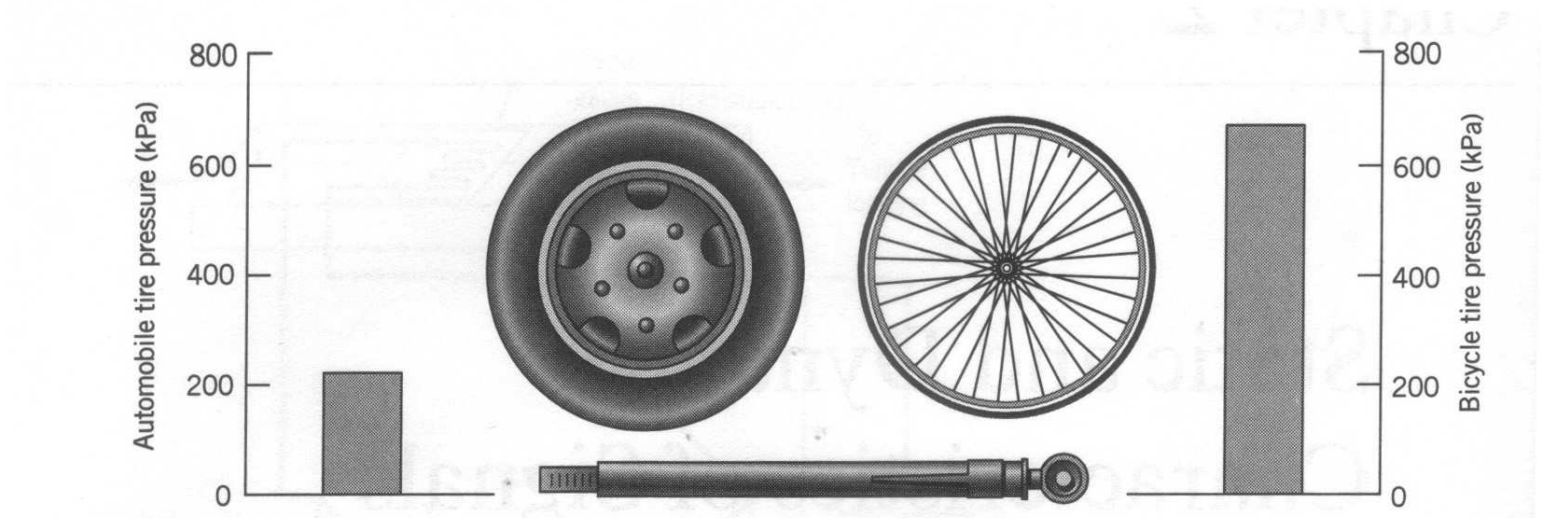
2141-375
Measurement and Instrumentation

Static and Dynamic Characteristics of Signal

Input-Output Signal Concepts

Engineering tasks in the measurement of physical variables
(1) Selection of a measurement system
(2) Interpreting the output from a measurement system.



Selection of Measurement system based on input signal range

Static signal: range
Time-varying signal: magnitude and the rate of change of a variable

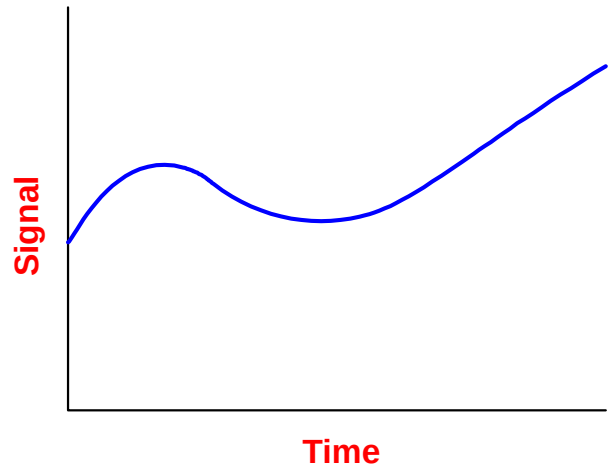
Signal Classification

- **Signal**: the physical information about a measured variable

Continuous time, continuous value

Define for each instant of time and its amplitude may vary continuously with time and assume any value

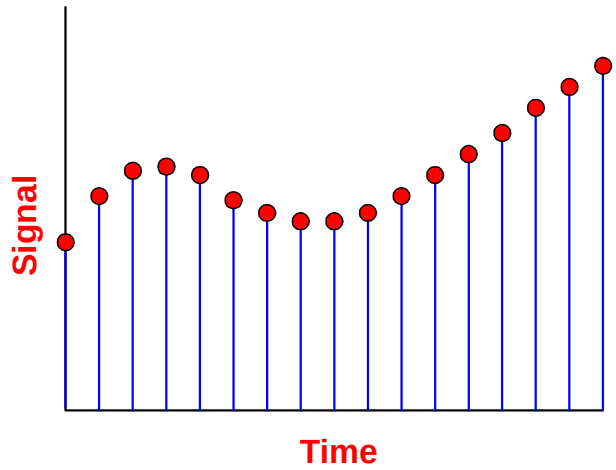
- Analog signal



Discrete time, continuous value

Define at discrete instants of time and its amplitude may vary continuously with time and assume any value

- Discrete time signal

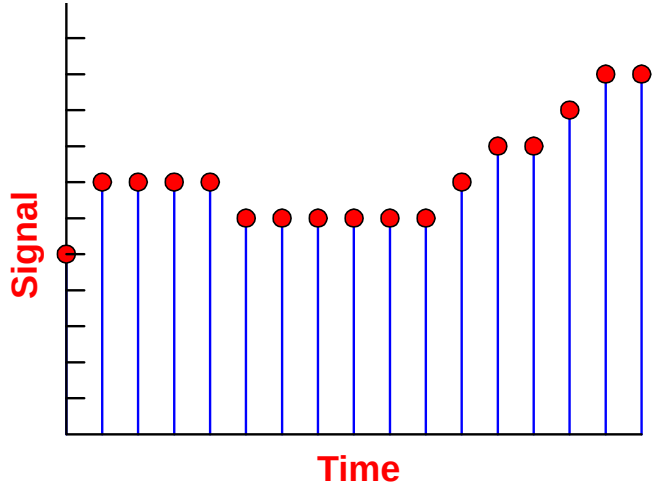


Signal Classification

Discrete time, discrete value

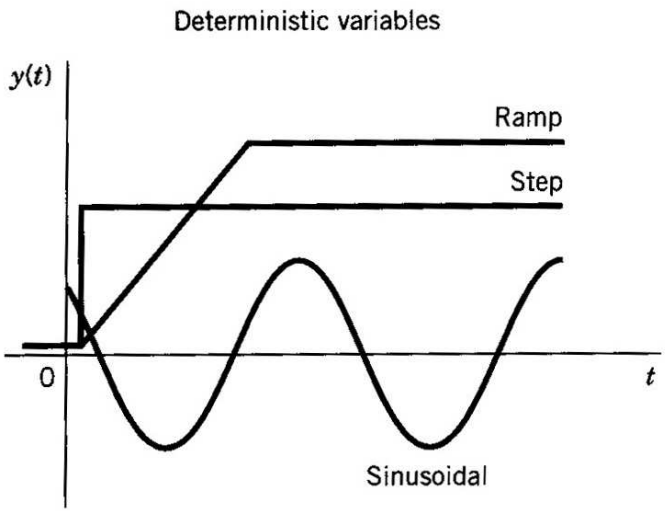
Define at discrete instants of time and
its amplitude may assume discrete value

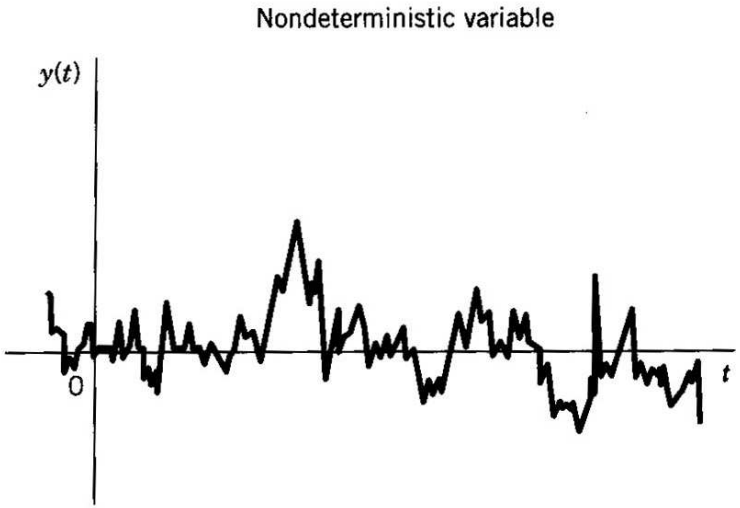
- Digital signal



Signal Classification: waveform

- **Static signal**: does not vary in time (or change very slowly).
- **Dynamic signal**: is defined as a time-dependent signal
- **Deterministic signal**: varies in time in a predictable manner.
  - **Periodic signal**: the variation of magnitude of the signal repeats at regular intervals
  - **Non-Periodic signal**: does not repeat at regular interval.
- **Non-Deterministic signal**: cannot be prescribed before it occurs.





Signal Classification: waveform

Table 2.1 Classification of Waveforms

I. Static

$$y(t) = A_0$$

II. Dynamic

Periodic waveforms

Simple periodic waveform

$$y(t) = A_0 + C\sin(\omega t + \phi)$$

Complex periodic waveform

$$y(t) = A_0 + \sum_{n=1}^{\infty} C_n \sin(n\omega t + \phi_n)$$

Aperiodic waveforms

Step[a]

$$y(t) = A_0 U(t) = A_0 \quad \text{for } t > 0$$

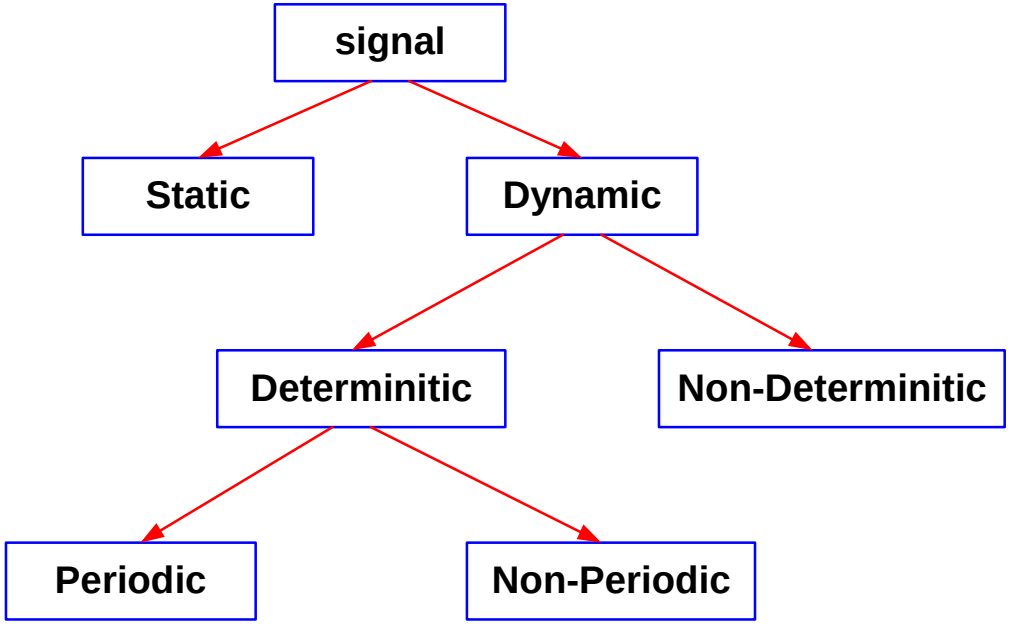
Ramp

$$y(t) = Kt \quad \text{for } 0 < t < t_f$$

Pulse[b]

$$y(t) = A_0 U(t) - A_0 U(t - t_1)$$

III. Nondeterministic waveform

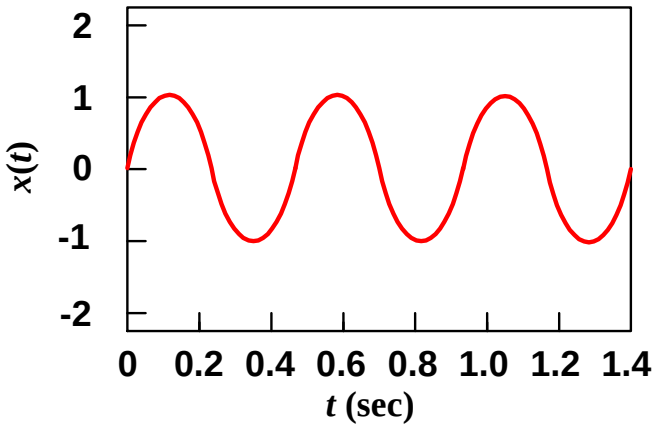


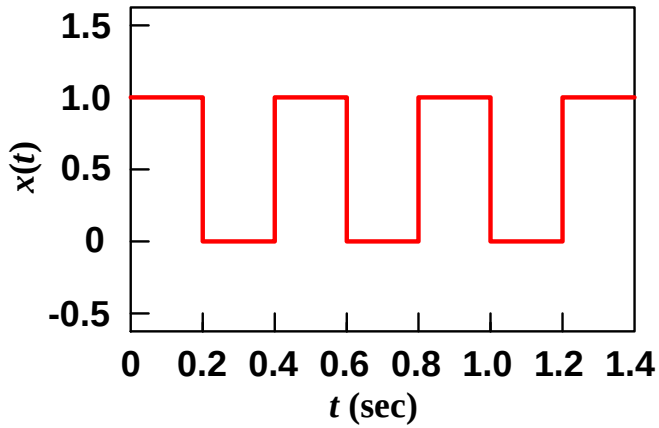
Periodic Signal

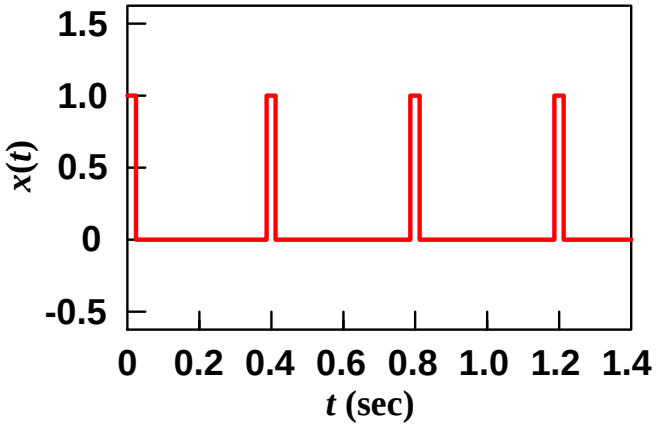
Mathematic of periodic signal

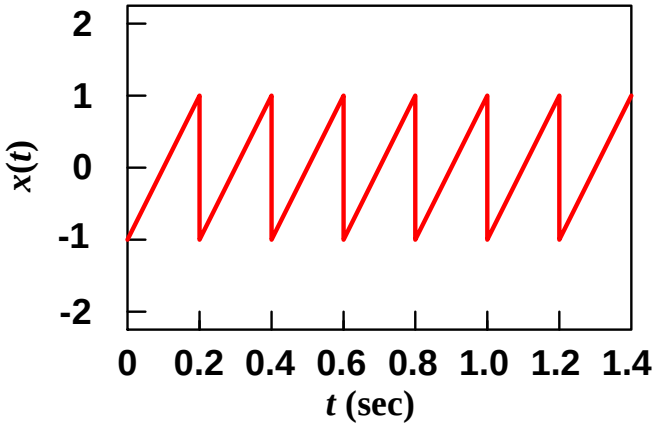
$$x(t) = x(t + nT), \quad \text{for} \quad n = 1,2,3$$

Where T is the period of the signal $x(t)$





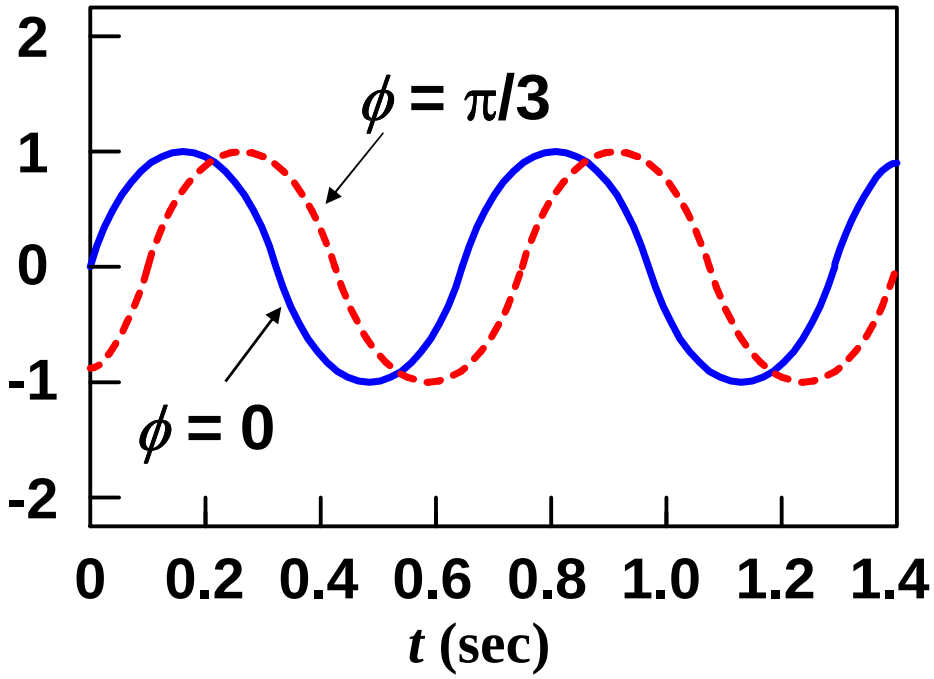




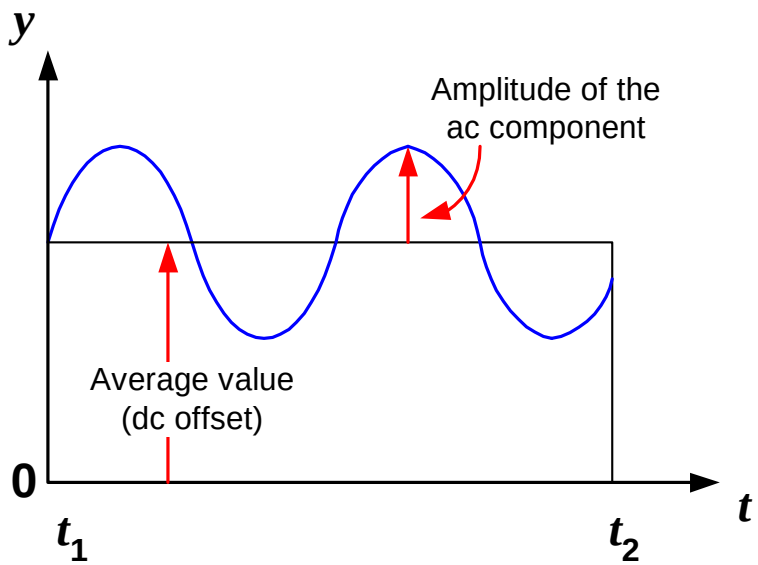
Sinusoidal Signal

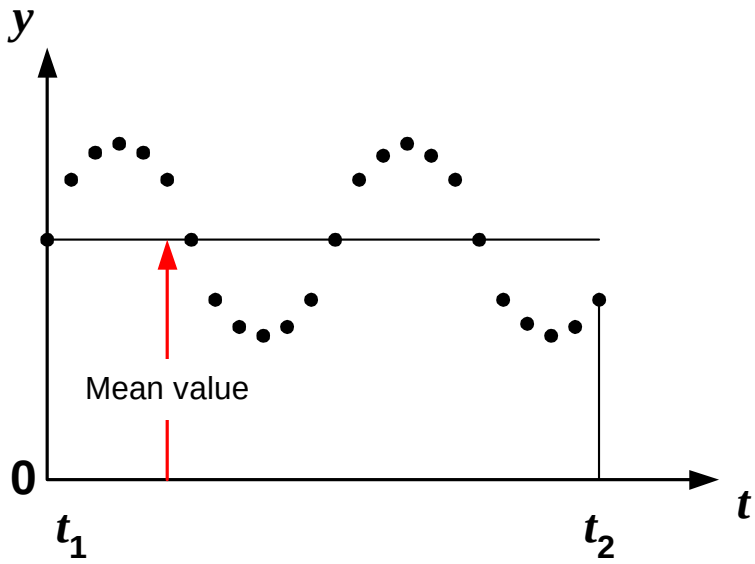
Generic sinusoidal signal: $x(t) = A\sin(\omega t + \phi)$

Where A is the amplitude, ω is the radian frequency and ϕ is the phase



Signal Analysis





- **Average or mean value during $t_2 - t_1$**

Analog

$$\bar{y} \equiv \frac{\int_{t_1}^{t_2} y(t)dt}{\int_{t_1}^{t_2} dt}$$

Discrete or digital

$$\bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i$$

- **root-mean-square value during $t_2 - t_1$**

Analog

$$y_{rms} \equiv \sqrt{\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} y(t)^2 dt}$$

Discrete or digital

$$y_{rms} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} y_i^2}$$

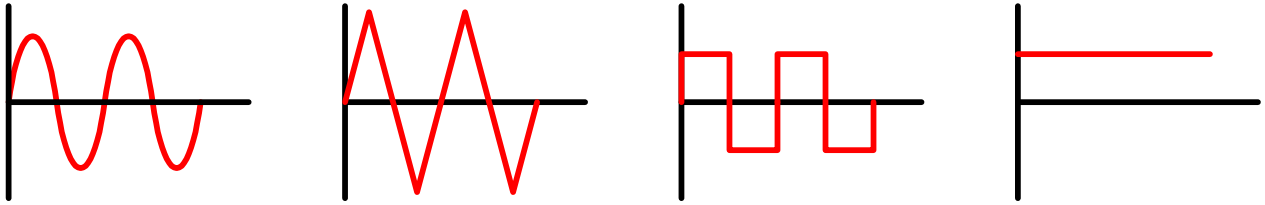
RMS: Root-Mean-Square

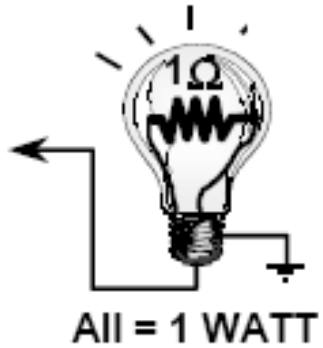
RMS is a measure of a signal's average power. Instantaneous power delivered to a resistor is $P = [v(t)]^2/R$.

$$P_{avg} = \frac{1}{R}\left(\frac{1}{T}\int_{t_0}^{t_0+T}[v(t)]^2\,dt\right) = \frac{(V_{rms})^2}{R} \quad \Rightarrow \quad V_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T}[v(t)]^2\,dt}$$

An AC voltage with a given RMS value has the same heating (power) effect as an DC voltage with the same value.

All the following voltage waveforms have the same RMS value, and should indicate 1 V_{ac} on an rms meter





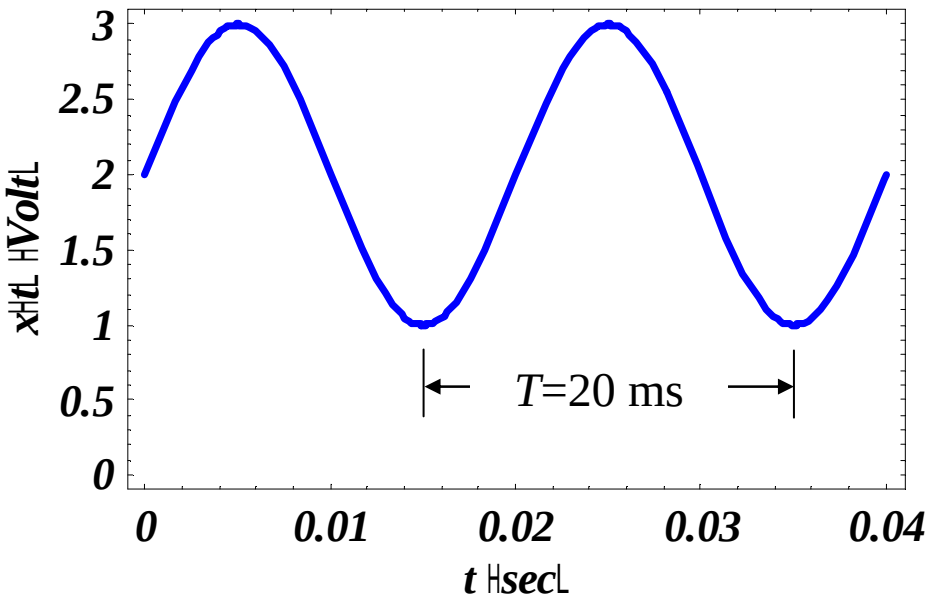
Waveform	Sine	Triangle	Square	DC
Vpeak	1.414	1.733	1	1
Vrms	1	1	1	1

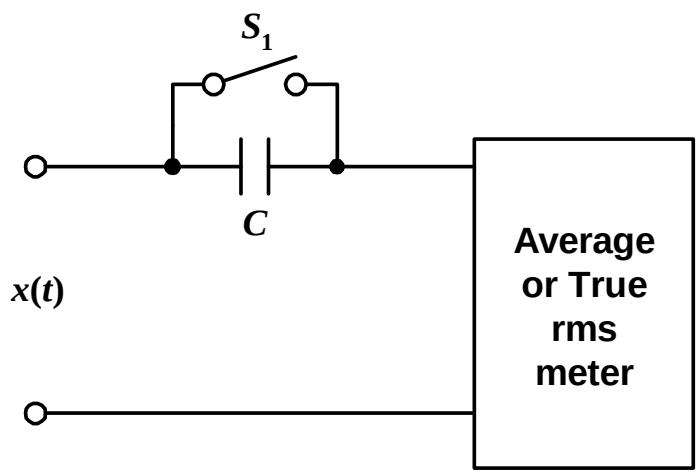
Signal Analysis

Ex Given a voltage signal as follows, the ac component has the frequency of 50 Hz, determine the average and rms values of signal that will be indicated by the meter when (a) S1 is open and (b) S1 is closed

Known signal characteristics

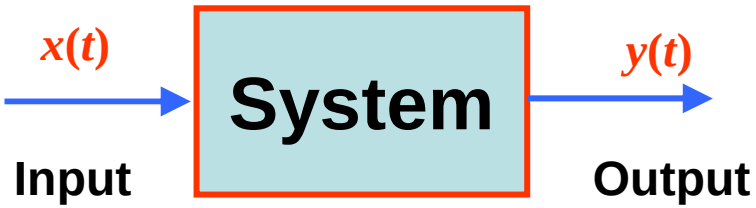
Find average and rms values

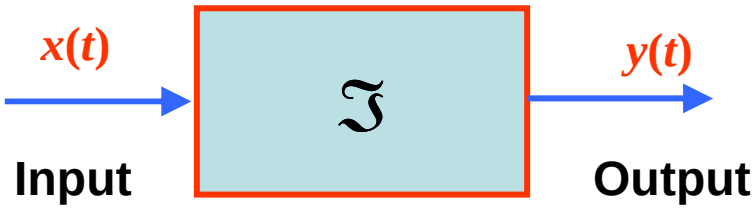




System

Signal are always associated with one or more systems. A system transforms the input into an output of the system.

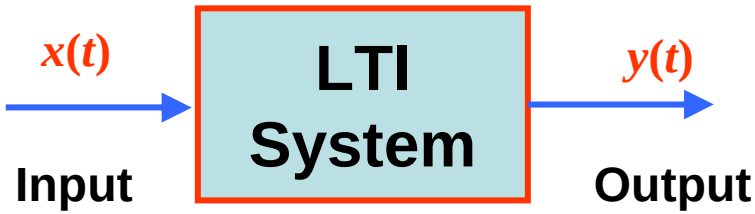




Block diagram of a system

$\Im$ is an mathematical operator

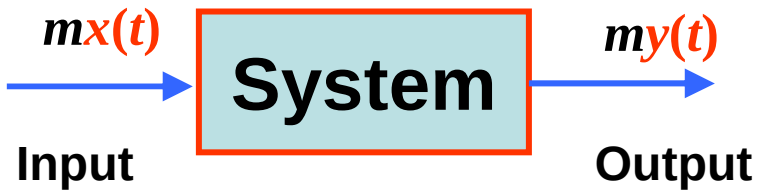
Linear, time invariant (LTI) system



A linear system is one which is both homogeneous (scale) and additive

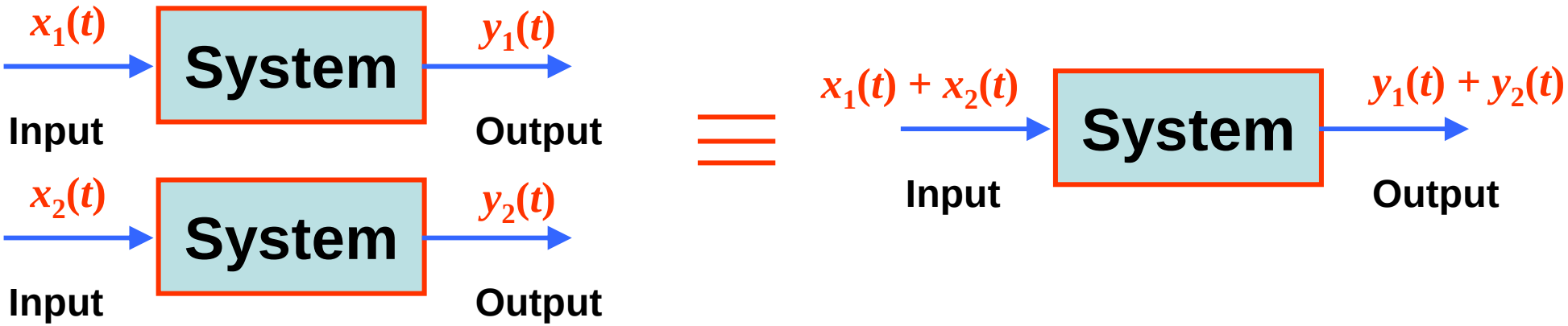
System

A homogeneous system is one which a scaled input produces an equally scaled output.



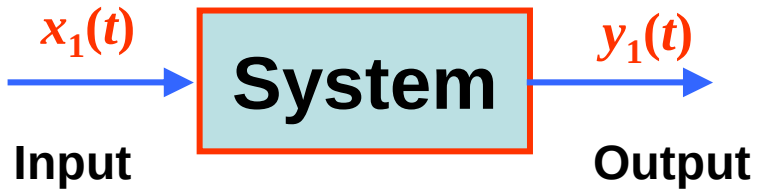
Homogeneous system

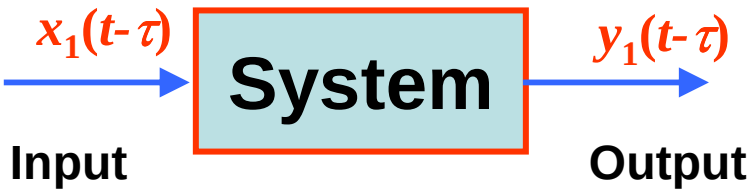
An additive system is one for which



System

A time invariant system is one for which a delay in the application of the input signal results in the same delay in the output.

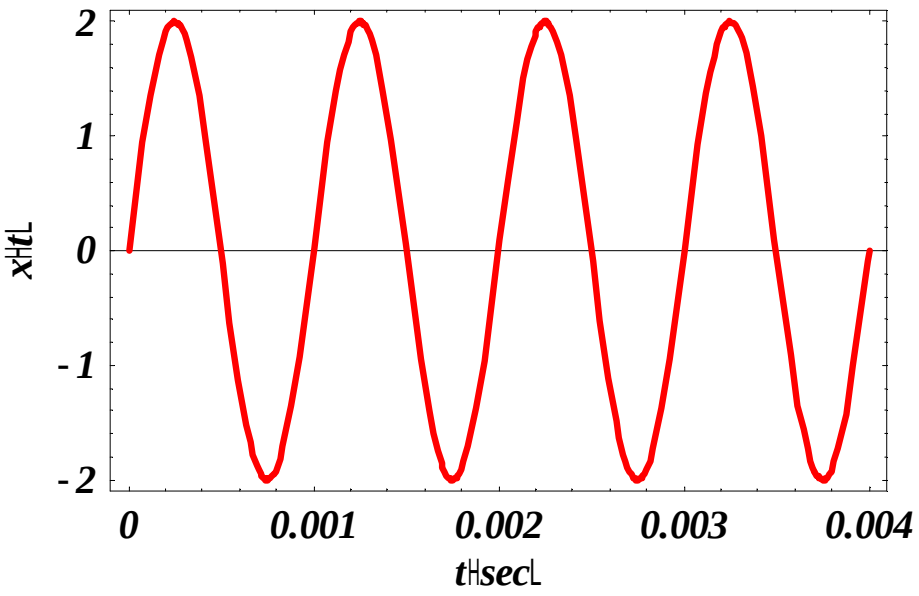


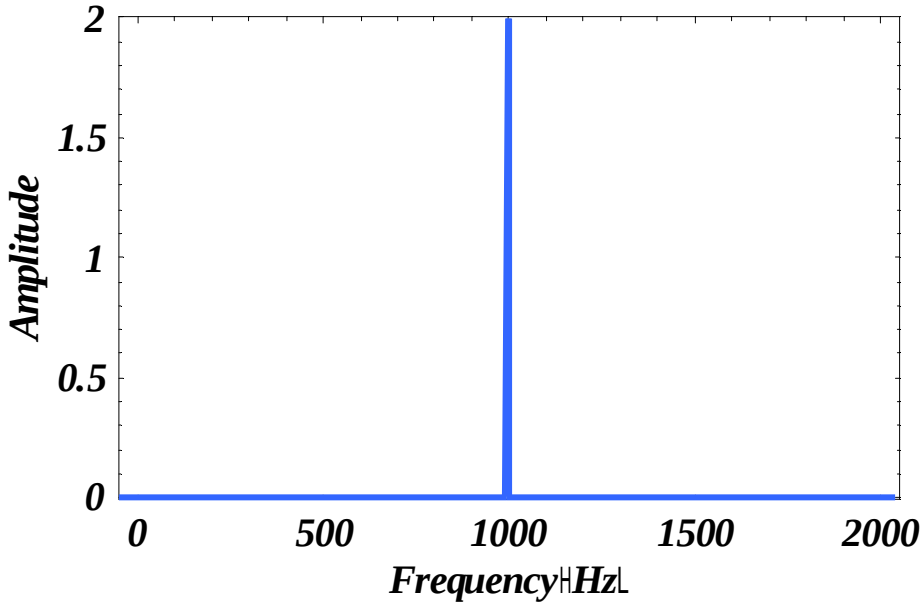


Time and Frequency Domain

A signal in frequency domain shows "How much" of a signal is associated with a certain frequency.

$$x(t) = 2\sin(2000\pi t)$$

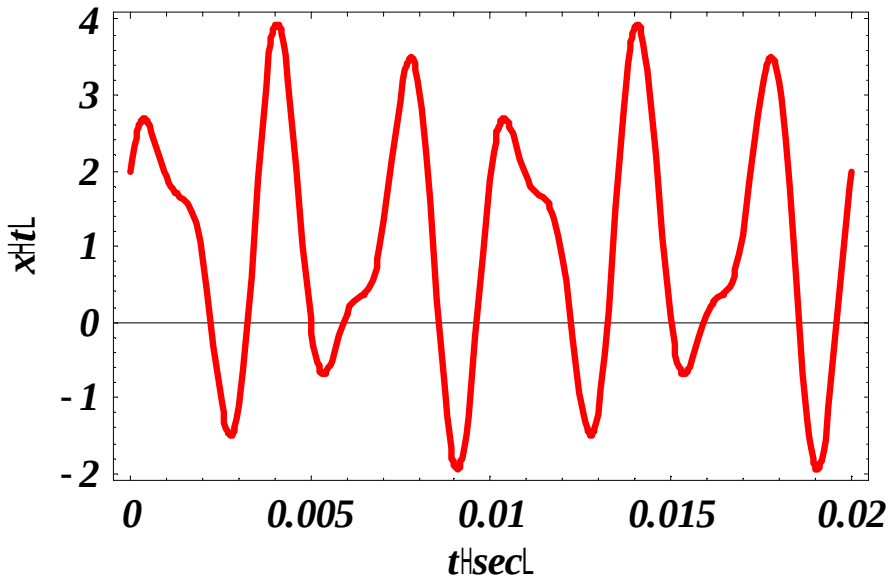


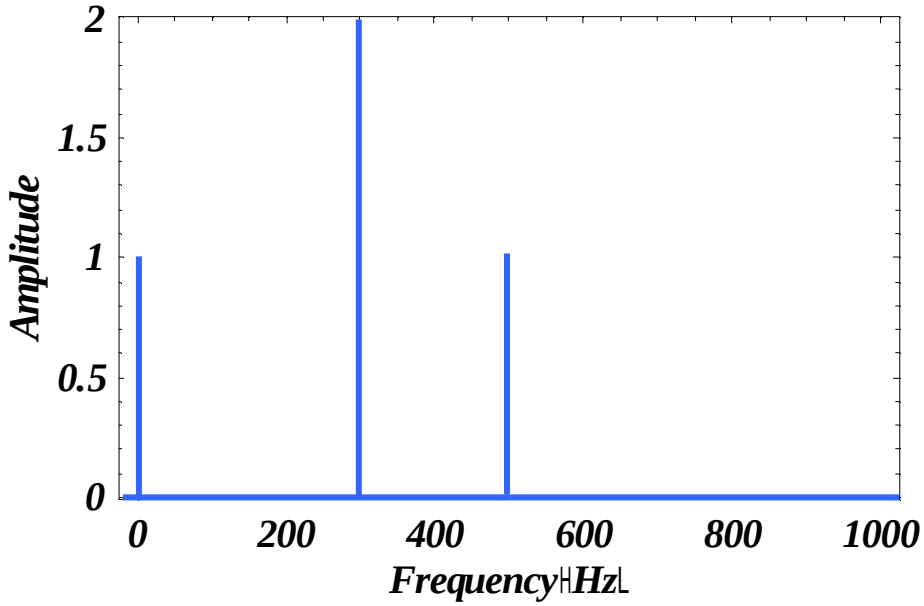


Time and frequency domain representation of the signal.

Time and Frequency Domain

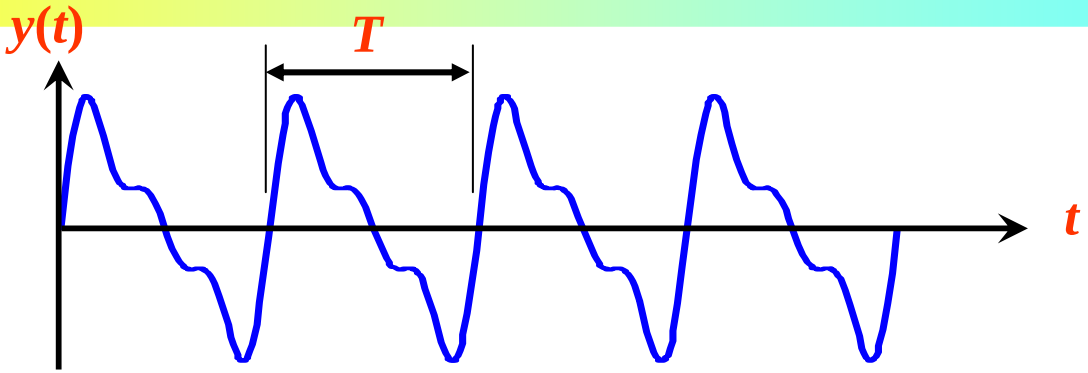
$$x(t) = 1 + 2\sin(600\pi t) + \cos(1000\pi t)$$





Comparing signals in the time and frequency domain

Fourier Series of Complex Periodic Signal



Fourier series for $y(t)$:

$$y(t) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos n\omega_0 t + B_n \sin n\omega_0 t \right)$$

Where the fundamental frequency $\omega_0 = \dfrac{2\pi}{T}$

A_0, A_n, B_n are constants that depend on $y(t)$ and n

$$A_0 = \frac{1}{T} \int_{-T/2}^{T/2} y(t)\,dt$$

$$A_n = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \cos n\omega_0 t\,dt; \quad n = 1,2,3,\ldots \qquad B_n = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \sin n\omega_0 t\,dt; \quad n = 1,2,3,\ldots$$

Useful Trigonometric Relations

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$
$$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$
$$\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$
$$\cos A - \cos B = 2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(B-A)$$

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$
$$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$

Useful Trigonometric Relations

Orthogonal principles

$$\frac{1}{T}\int_0^T \sin\left(2\pi m\frac{t}{T}\right)\sin\left(2\pi m\frac{t}{T}\right)dt = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \\ 0 & m = n = 0 \end{cases}$$

$$\frac{1}{T}\int_0^T \cos\left(2\pi m\frac{t}{T}\right)\cos\left(2\pi n\frac{t}{T}\right)dt = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \\ 1 & m = n = 0 \end{cases}$$

$$\frac{1}{T}\int_0^T \sin\left(2\pi m\frac{t}{T}\right)\cos\left(2\pi n\frac{t}{T}\right)dt = 0$$

Fourier Series of Periodic Signal

A function $f(t)$ is even if it is symmetric about the vertical axis

$$f(t) = f(-t)$$

A function $f(t)$ is odd if it is symmetric about the origin

$$f(t) = -f(-t)$$

Cosine is even function and Sine is odd function

Fourier Cosine Series $$y(t) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos n\omega_0 t \right)$$

Fourier Sine Series $$y(t) = A_0 + \sum_{n=1}^{\infty} \left(B_n \sin n\omega_0 t \right)$$

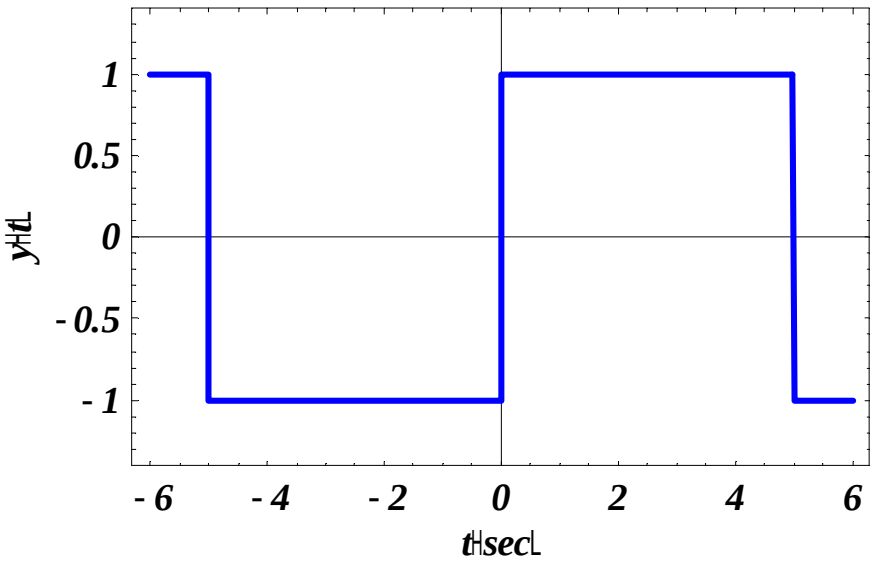
Fourier Series of Periodic Signal

Ex Determine the Fourier series that represents the function as shown below

Known T = 10 and A_0 = 0 (Symmetrical)

Find Fourier coefficient A_n and B_n

Solution Since the function is odd, the Fourier series will contain only sine terms



$$y(t) = \sum_{n=1}^{\infty} B_n \sin \frac{2n\pi t}{T}$$

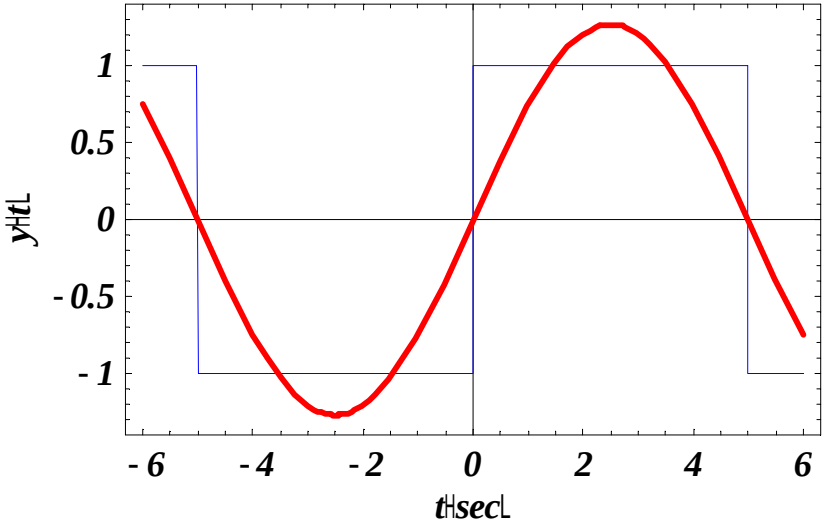
$$B_n = \frac{4}{n\pi}; \qquad n = \text{odd number}$$

The resulting Fourier Series is then

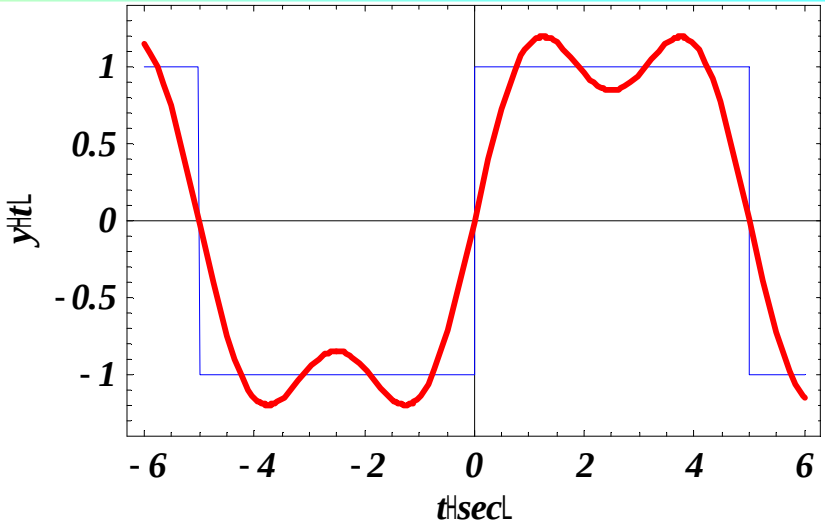
$$y(t) = \frac{4}{\pi}\sin\frac{2\pi}{10}t + \frac{4}{3\pi}\sin\frac{6\pi}{10}t + \frac{4}{5\pi}\sin\frac{10\pi}{10}t + \ldots$$

Fourier Series of Periodic Signal

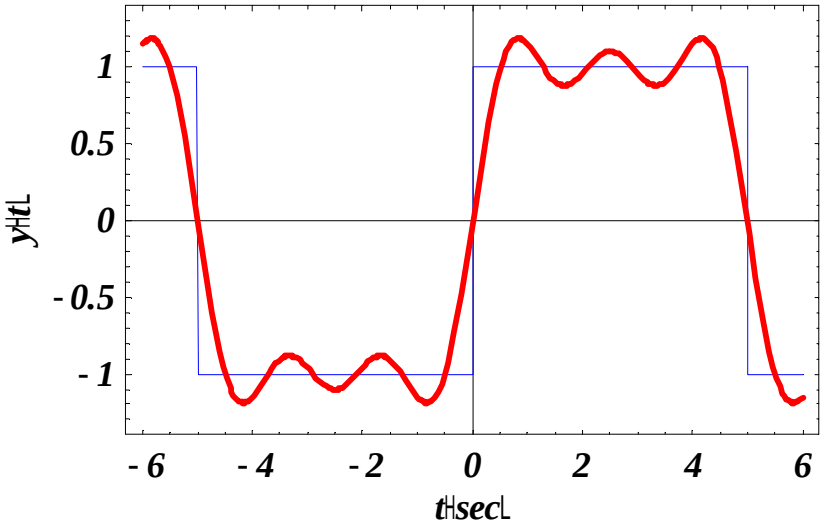
$$y(t) = \frac{4}{\pi}\sin\frac{2\pi}{10}t$$



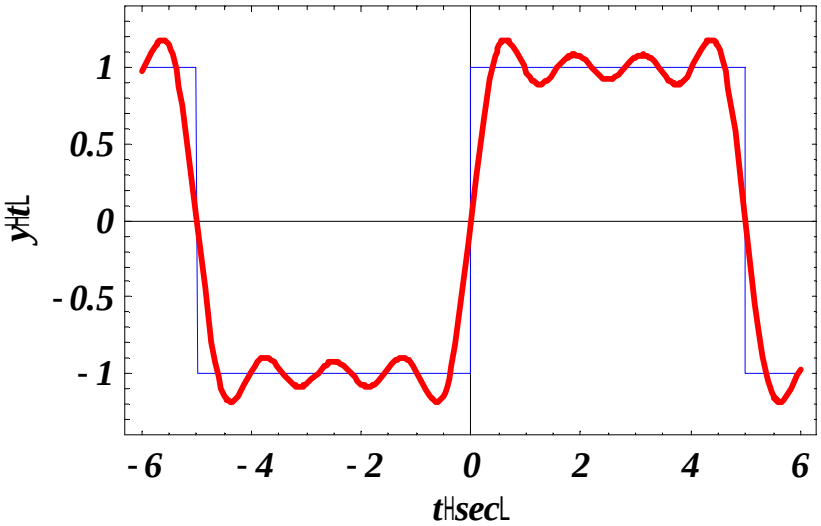
$$y(t) = \frac{4}{\pi}\sin\frac{2\pi}{10}t + \frac{4}{3\pi}\sin\frac{6\pi}{10}t$$



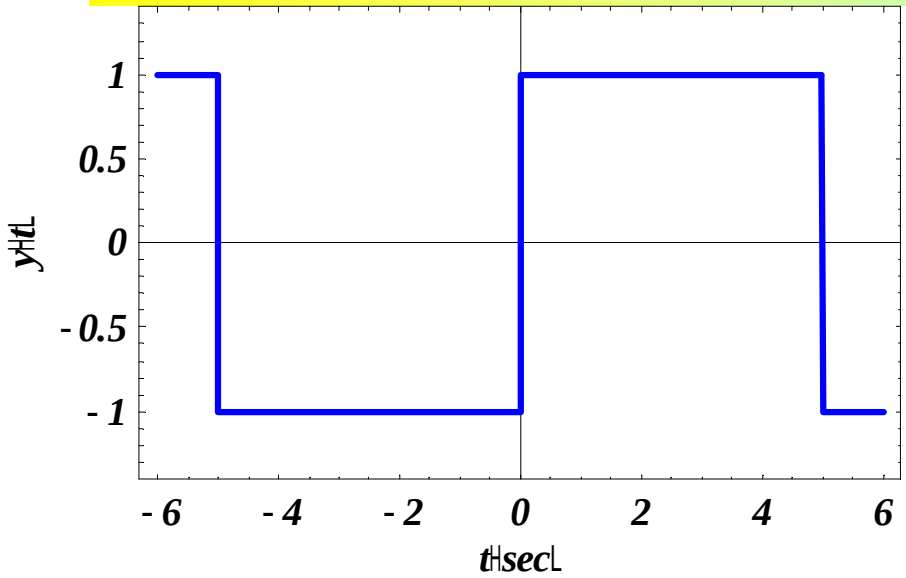
$$y(t) = \frac{4}{\pi}\sin\frac{2\pi}{10}t + \frac{4}{3\pi}\sin\frac{6\pi}{10}t + \frac{4}{5\pi}\sin\frac{10\pi}{10}t$$

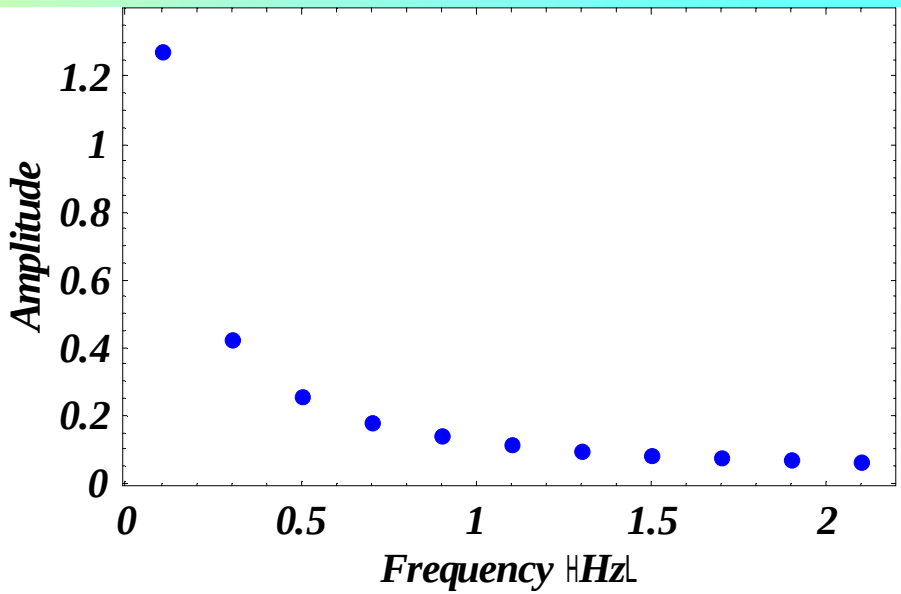


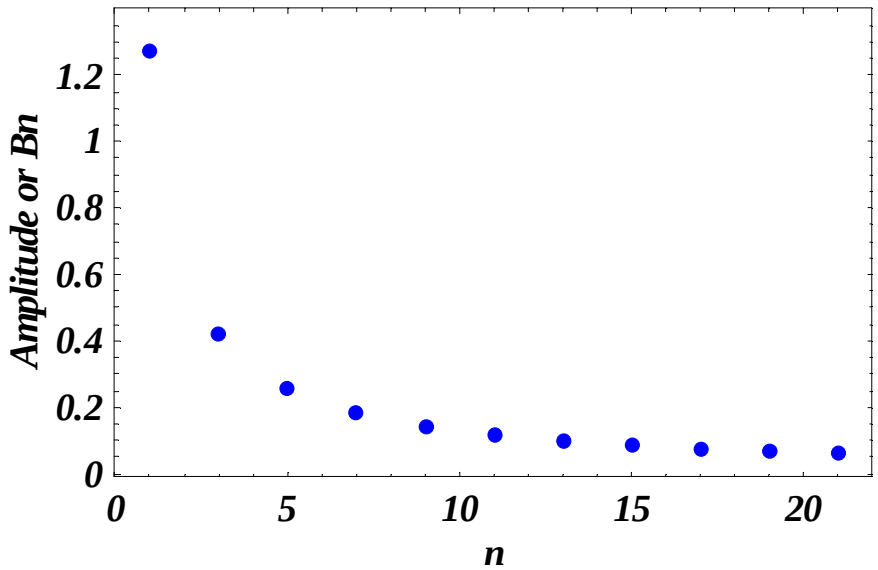
$$y(t) = \frac{4}{\pi}\sin\frac{2\pi}{10}t + \frac{4}{3\pi}\sin\frac{6\pi}{10}t + \frac{4}{5\pi}\sin\frac{10\pi}{10}t + \frac{4}{7\pi}\sin\frac{14\pi}{10}t$$



Fourier Series of Periodic Signal







Fourier Series of Periodic Signal

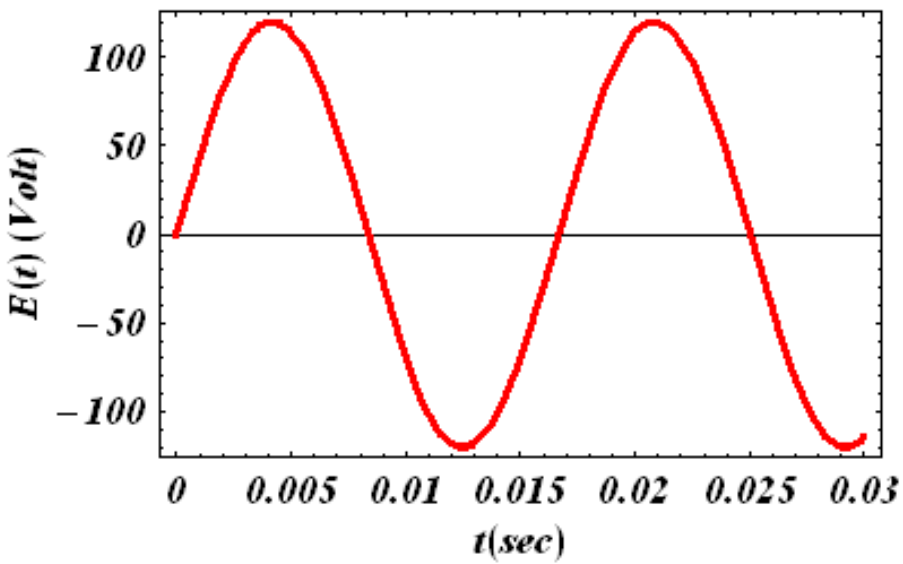
Ex find the frequency content of the output voltage from a full-wave rectifier, If the ac input signal is given by

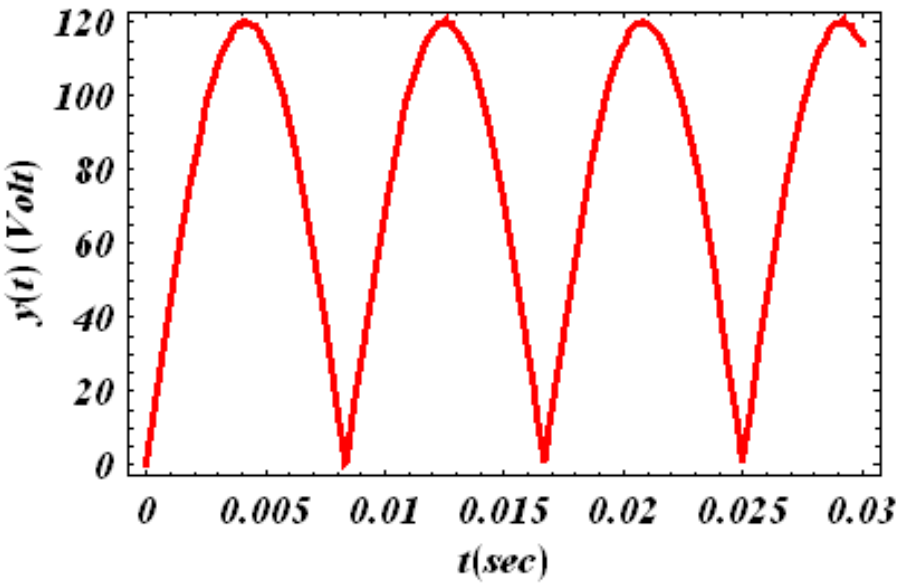
$$E(t) = 120 \sin 120\pi t$$

Known The full-wave signal can be expressed $y(t) = |120 \sin 120\pi t|$

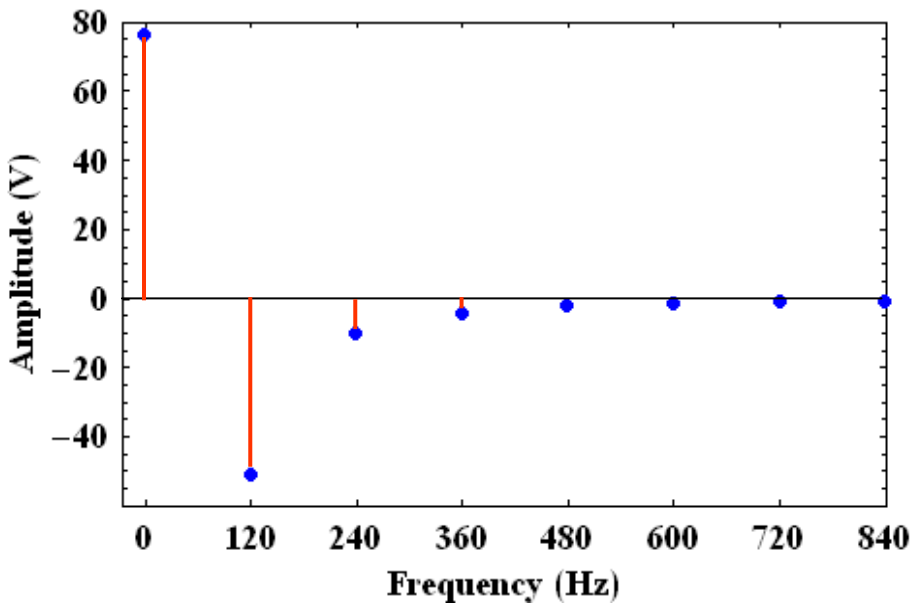
Find The frequency content of this signal

Solution Since the function is even, the Fourier series will contain only cos terms





Fourier Series of Periodic Signal



$$y(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{2n\pi t}{T}$$

$$A_0 = \frac{2 \times 120}{\pi}$$

$$A_n = \frac{120}{\pi}\left(\frac{-2}{n-1} + \frac{2}{n+1}\right); \qquad n = \text{even number}$$

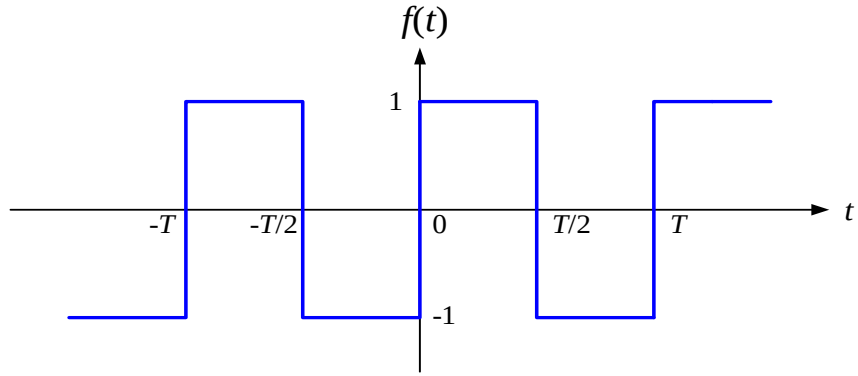
The resulting Fourier Series is then

$$y(t) = \frac{2 \times 120}{\pi} + \frac{120}{\pi} \sum_{n} \frac{4}{1-n^2} \cos 2\pi n 60 t$$

$n = \text{even number}$

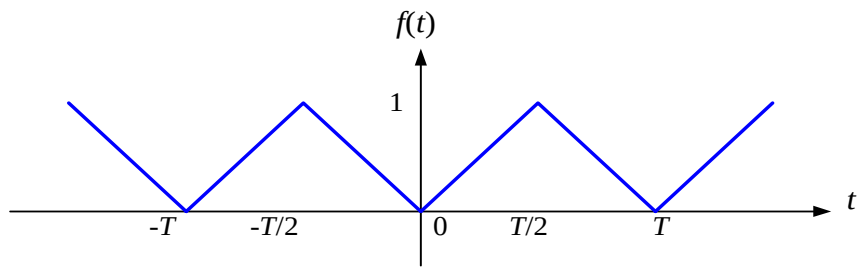
$$f(t)=\begin{cases}1 & 0<t<\dfrac{T}{2}\\ -1 & -\dfrac{T}{2}<t<0\end{cases}$$

$$f(t)=\frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sin\frac{2n\pi}{T}t\right)\qquad n=\text{odd number}$$



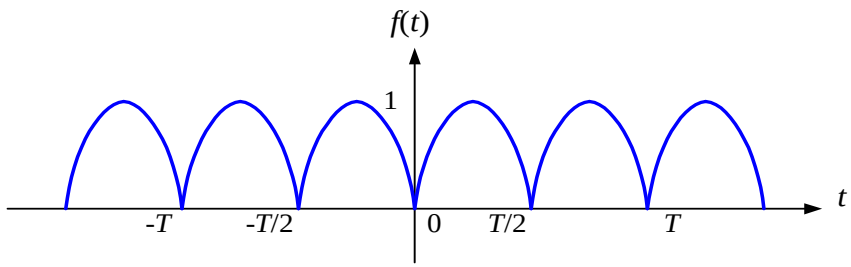
$$f(t)=\left|\frac{2}{T}t\right|=\begin{cases}\dfrac{2}{T}t & 0<t<\dfrac{T}{2}\\ -\dfrac{2}{T}t & -\dfrac{T}{2}<t<0\end{cases}$$

$$f(t)=\frac{1}{2}-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos\frac{2n\pi}{T}t\right)\qquad n=\text{odd number}$$



$$f(t)=\left|\sin\frac{2\pi}{T}t\right|=\begin{cases}\sin\dfrac{2\pi}{T}t & 0<t<\dfrac{T}{2}\\ -\sin\dfrac{2\pi}{T}t & -\dfrac{T}{2}<t<0\end{cases}$$

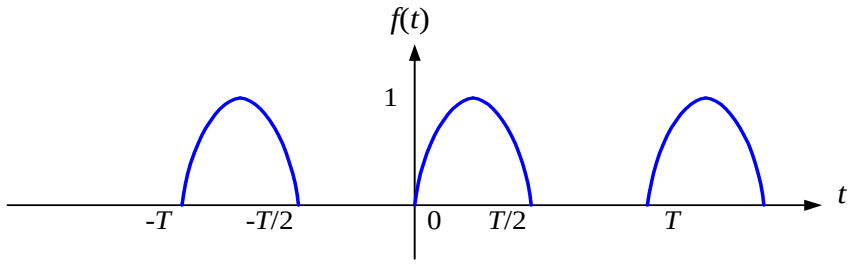
$$f(t)=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=2}^{\infty}\left(\frac{1}{\left(1-n^2\right)}\cos\frac{2n\pi}{T}t\right)\qquad n=\text{even number}$$



$$f(t)=\begin{cases}\sin\dfrac{2\pi}{T}t & 0<t<\dfrac{T}{2}\\ 0 & -\dfrac{T}{2}<t<0\end{cases}$$

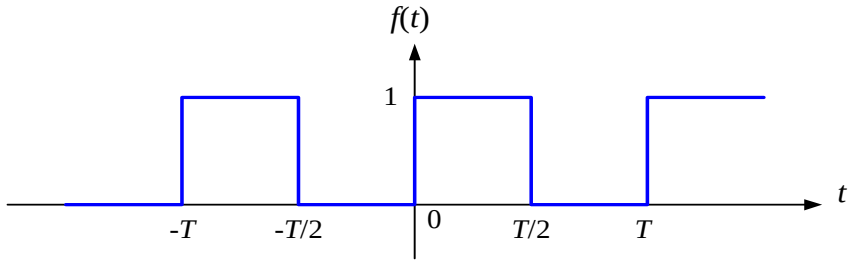
$$f(t)=\frac{1}{\pi}+\frac{1}{2}\sin\frac{2\pi}{T}t+\frac{2}{\pi}\sum_{n=2}^{\infty}\left(\frac{1}{\left(1-n^2\right)}\cos\frac{2n\pi}{T}t\right)$$

$n=$ even number



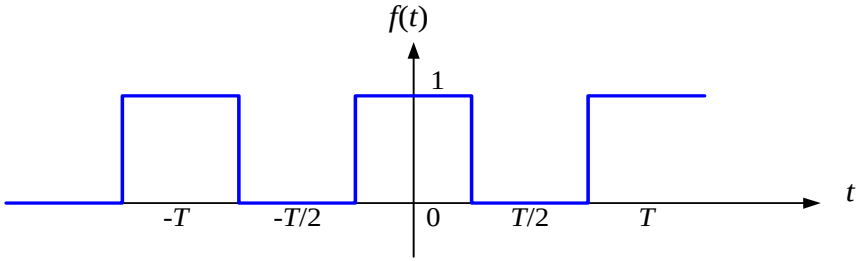
$$f(t)=\begin{cases}1 & 0<t<\dfrac{T}{2}\\ 0 & -\dfrac{T}{2}<t<0\end{cases}$$

$$f(t)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sin\frac{2n\pi}{T}t\right)\qquad n=\text{odd number}$$



$$f(t)=\begin{cases}1 & -\dfrac{T}{4}<t<\dfrac{T}{4}\\ 0 & -\dfrac{T}{2}<t<-\dfrac{T}{4}\ \text{and}\ \dfrac{T}{4}<t<\dfrac{T}{2}\end{cases}$$

$$f(t)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{\sin n\pi}{n}\cos\frac{2n\pi}{T}t\right)\qquad n=\text{odd number}$$



Fourier Transform of Complex Signal

The Fourier transform is a mathematical function that transforms a signal from the time domain, $y(t)$ to the frequency domain, $Y(f)$.

Definition of Fourier Transform of $y(t)$

$$Y(f) = \int_{-\infty}^{\infty} y(t)\, e^{-i2\pi ft}\, dt$$

Definition of Inverse Fourier Transform of $Y(f)$

$$y(t) = \int_{-\infty}^{\infty} Y(f)\, e^{i2\pi ft}\, df$$

Time Domain		*Frequency Domain*
$y(t)$	$\xleftrightarrow{\Im}$	$Y(f)$

Fourier Transform

Ex Determine the Fourier transform, consider the one-sided exponential signal.

$$x(t) = e^{-7t}u(t)$$

Known $x(t) = e^{-7t}u(t)$

Find Fourier Transform of $x(t)$

Solution

$$X(f) = \Im[x(t)] = \int_0^{\infty} e^{-7t} e^{-j2\pi ft} dt$$

$$\boxed{\Im[u(t)e^{-7t}] = \frac{1}{7 + j2\pi f}}$$

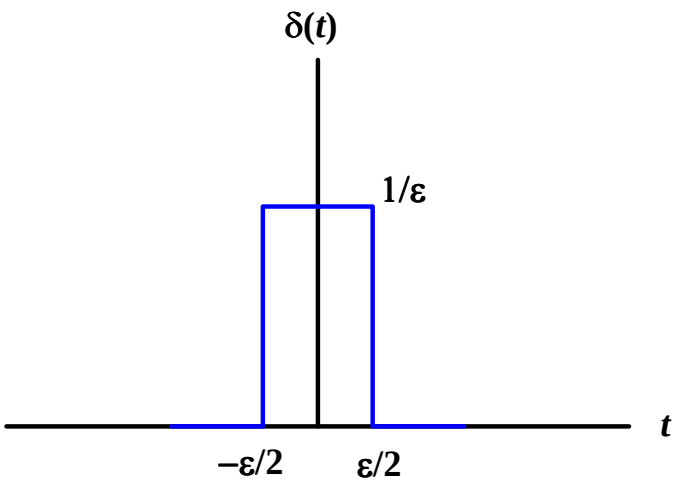
Impulse Function: Review

$$\delta_\varepsilon(t) = \begin{cases} 1/\varepsilon; & -\varepsilon/2 < t < \varepsilon/2 \\ 0; & \text{elsewhere} \end{cases}$$

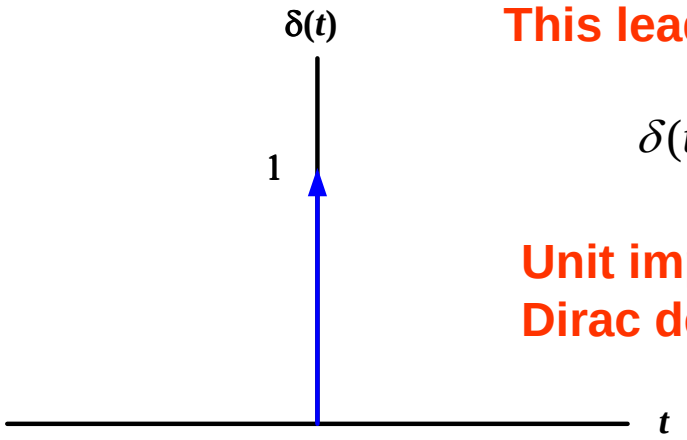
Which implies

$$\int_{-\infty}^{+\infty} \delta_\varepsilon(t)dt = 1$$

The function may be considered as a rectangular pulse of width ε and height $1/\varepsilon$. In the limit $\varepsilon \to 0$, the height increase in such a way that the area is 1



Visualization



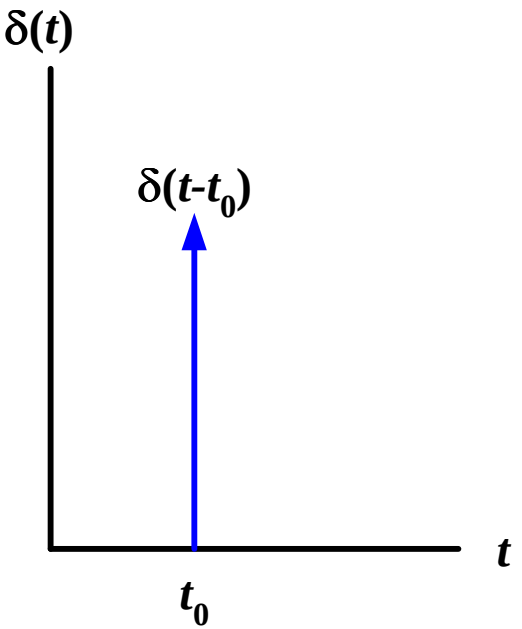
symbol

This leads to the definition

$$\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$$

Unit impulse function or Dirac delta function

Impulse Function: Review



$$\delta(t-t_0)=\begin{cases}\infty; & t=t_0\\ 0; & \text{elsewhere}\end{cases}$$

Shifting of function by t_0

$$\left.\begin{aligned}\delta(t)&=\int_{-\infty}^{+\infty}e^{j2\pi ft}df\\ \delta(t-t_0)&=\int_{-\infty}^{+\infty}e^{j2\pi f(t-t_0)}df\end{aligned}\right\}$$

$$\delta(f)=\int_{-\infty}^{+\infty}e^{-j2\pi ft}dt$$

$$\delta(f-f_0)=\int_{-\infty}^{+\infty}e^{-j2\pi(f-f_0)t}dt$$

$$\int_{-\infty}^{+\infty}f(t)\delta(t-t_0)dt=f(t_0)$$

$$\int_{-\infty}^{+\infty}\delta(t)e^{-j2\pi ft}dt=1$$

Discrete Fourier Transform

Discrete Fourier Transform (DFT) of y_r

$$y_r = y(r\delta t) \qquad r = 1,2,\ldots,N$$

$$Y(f_k) = \frac{2}{N}\sum_{r=1}^{N} y(r\delta t)e^{-i2\pi rk/N} \qquad k = 1,2,\ldots,\frac{N}{2}$$

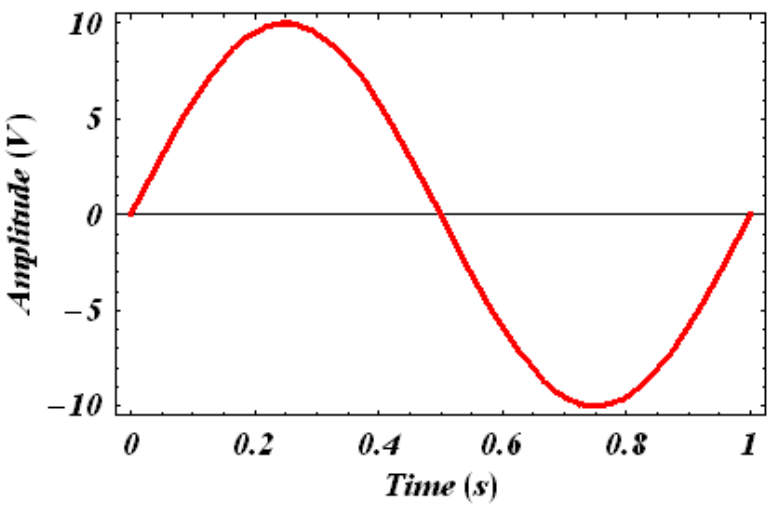
where $f_k = k\delta f$ **and** $\delta f = \dfrac{1}{N\delta t} = \dfrac{f_s}{N}$

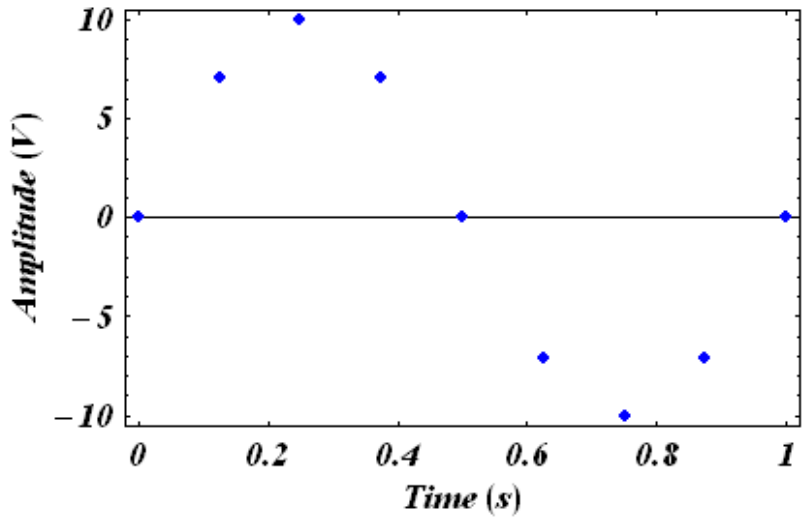
Discrete Fourier Transform

Example: Estimate the amplitude spectrum or frequency content of the discrete data taken from $y(t) = 10 \sin 2\pi t$ using a time increment of 0.125 s for the duration of 1 s.

Known: $\delta t = 0.125$ s or $f_s = 8$ Hz

Solution:





Discrete Fourier Transform

Discrete Data Set for $y(t) = 10 \sin 2\pi t$

r	$y(r\,\delta t)$
1	7.071
2	10.000
3	7.071
4	0.000
5	-7.071
6	-10.000
7	-7.071
8	0.000

$$\delta f = \frac{f_s}{N} = \frac{8}{8} = 1\,\text{Hz}$$

$$f_k = k\delta f = k\ \text{Hz}$$

Discrete Fourier Transform of $y(t)$

k	f_k (Hz)	$Y(f_k)$	$\lvert Y(f_k)\rvert$
1	1	-10i	10
2	2	0	0
3	3	0	0
4	4	0	0