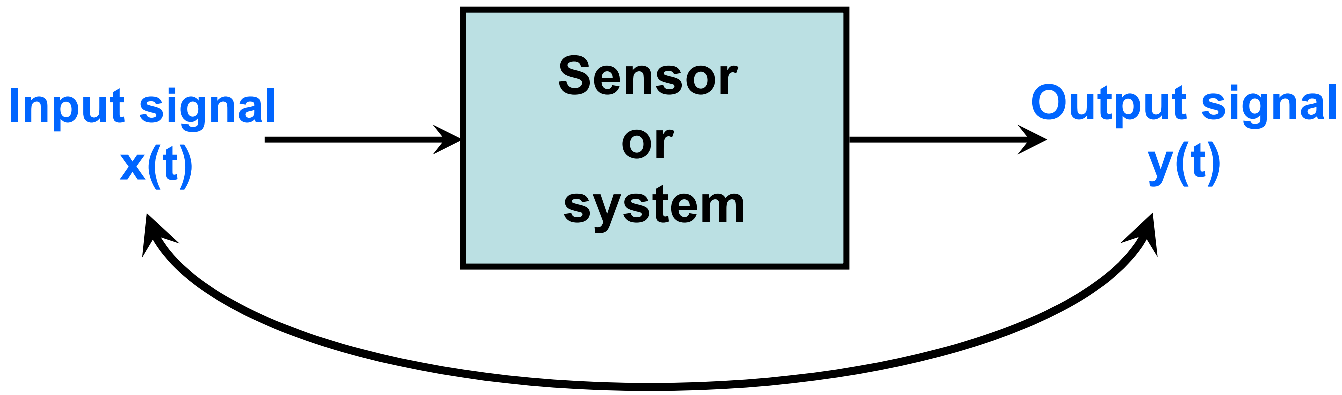
Dynamic Characteristics

Dynamic characteristics tell us about how well a sensor responds to changes in its input. For dynamic signals, the sensor or the measurement system must be able to respond fast enough to keep up with the input signals.



In many situations, we must use y(t) to infer x(t), therefore a qualitative understanding of the operation that the sensor or measurement system performs is imperative to understanding the input signal correctly.

General Model For A Measurement System

n^{th} Order ordinary linear differential equation with constant coefficient

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = \underbrace{b_m \frac{d^m x(t)}{dt^m} + b_{m-1}\frac{d^{m-1} x(t)}{dt^{m-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)}_{F(t)\text{ = forcing function}}$$

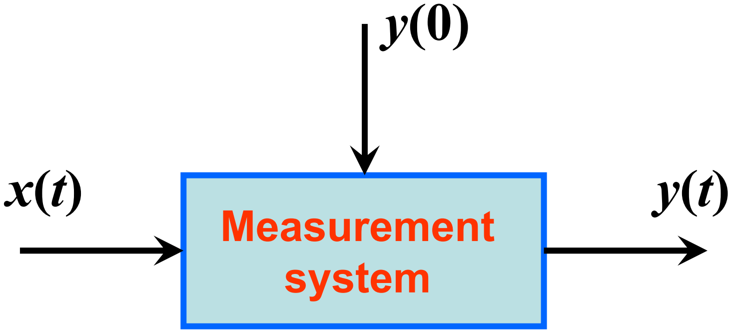
Where $m \le n$

$y(t)$ = output from the system

$x(t)$ = input to the system

t = time

a's and b's = system physical parameters, assumed constant



The solution $$y(t) = y_{ocf} + y_{opi}$$

Where y_{ocf} = complementary-function part of solution

y_{opi} = particular-integral part of solution

Complementary-Function Solution

The solution y_{ocf} is obtained by calculating the n roots of the algebraic *characteristic equation*

Characteristic equation

$$a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0 = 0$$

Roots of the characteristic equation:

$$D = s_1, s_2, \ldots, s_n$$

Complementary-function solution:

1. Real roots, unrepeated:

$$Ce^{st}$$

2. Real roots, repeated:
   each root s which appear p times

$$\left(C_0 + C_1 t + C_2 t^2 + \ldots + C_{p-1} t^{p-1}\right) e^{st}$$

3. Complex roots, unrepeated:
   the complex form: $a \pm ib$

$$Ce^{at} \sin(bt + \phi)$$

4. Complex roots, repeated:
   each pair of complex root which appear p times

$$[C_0 \sin(bt + \phi_0) + C_1 t \sin(bt + \phi_1) + C_2 t^2 \sin(bt + \phi_2) + \ldots + C_{p-1} t^{p-1} \sin(bt + \phi_{p-1})] e^{at}$$

Particular Solution

Method of undetermined coefficients:

$$y_{opi} = Af(t) + Bf'(t) + Cf''(t) + \ldots$$

Where $f(t)$ = the function that describes input quantity

A, B, C = constant which can be found by substituting y_{opi} into ODEs

Important Notes

✓ • After a certain-order derivative, all higher derivatives are zero.

✓ • After a certain-order derivative, all higher derivatives have the same functional form as some lower-order derivatives.

✗ • Upon repeated differentiation, new functional forms continue to arise.

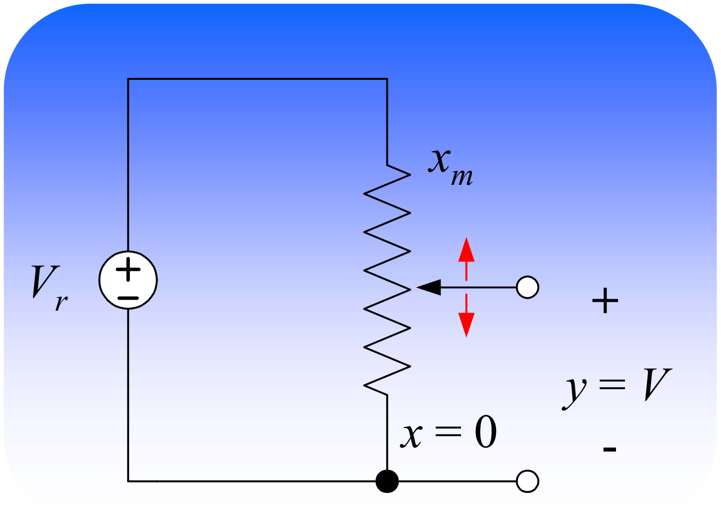
Zero-order Systems

All the a's and b's other than a_0 and b_0 are zero.

$$a_0 y(t) = b_0 x(t) \longrightarrow y(t) = Kx(t)$$

where K = static sensitivity = b_0/a_0

The behavior is characterized by its static sensitivity, K and remains constant regardless of input frequency (ideal dynamic characteristic).



$$V = V_r \cdot \frac{x}{x_m} \text{ here, } K = V_r / x_m$$

Where $0 \le x \le x_m$ and V_r is a reference voltage

A linear potentiometer used as position sensor is a zero-order sensor.

First-Order Systems

All the a's and b's other than a_1, a_0 and b_0 are zero.

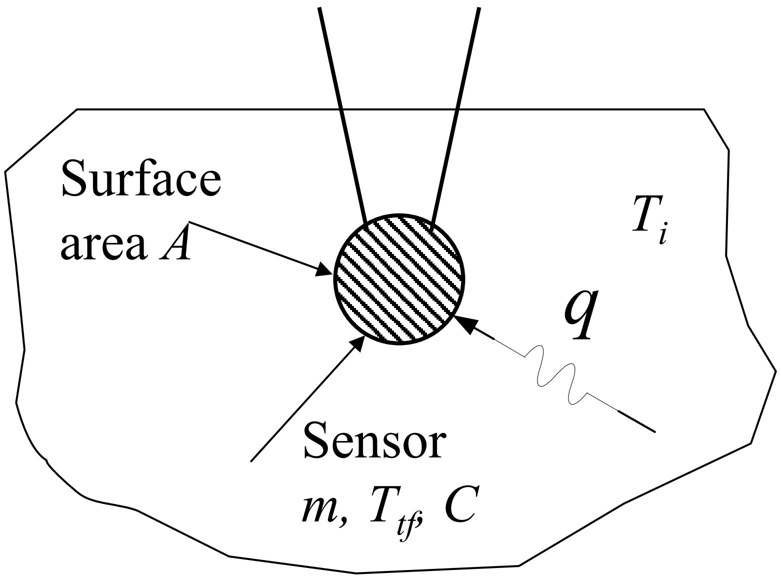
$$a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t)$$

$$\boxed{\tau \frac{dy(t)}{dt} + y(t) = Kx(t)} \Longleftrightarrow \boxed{\frac{y}{x}(D) = \frac{K}{\tau D + 1}}$$

Where $K = b_0/a_0$ is the static sensitivity

$\tau = a_1/a_0$ is the system's time constant (dimension of time)

First-Order Systems



Thermometer based on a mass, m with specified heat, C

Consider a thermometer based on a mass $m = \rho V$ with specified heat C (J/kg·K), heat transmission area A, and (convection heat transfer coefficient U (W/m^2·K).

(Heat in) – (Heat out) = Energy stored

Assume no heat loss from the thermometer

$$UA(T_i - T_{tf})dt - 0 = \rho VCdT_{tf}$$

$$\rho VC\frac{dT_{tf}}{dt} + UAT_{tf} = UAT_i$$

Therefore, we can immediately define K =1 and τ = $\rho VC/UA$

$$\tau\frac{dT_{if}}{dt} + T_{if} = T_i$$

First-Order Systems: Step Response

Assume for $t < 0$, $y = y_0$, at time = 0 the input quantity, x increases instantly by an amount A. Therefore $t > 0$

$$x(t) = AU(t) = \begin{cases} 0 & t \le 0 \\ A & t > 0 \end{cases}$$

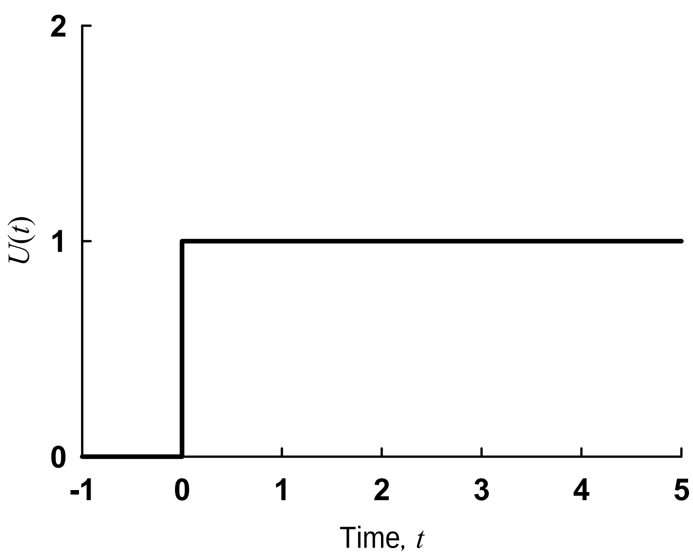
$$\tau \frac{dy(t)}{dt} + y(t) = KAU(t)$$

The complete solution:

$$\boxed{y(t) = Ce^{-t/\tau} + KA}$$

$$\underbrace{Ce^{-t/\tau}}_{y_{ocf}} \qquad \underbrace{KA}_{y_{opi}}$$

Transient response (y_{ocf}) **Steady state response** (y_{opi})



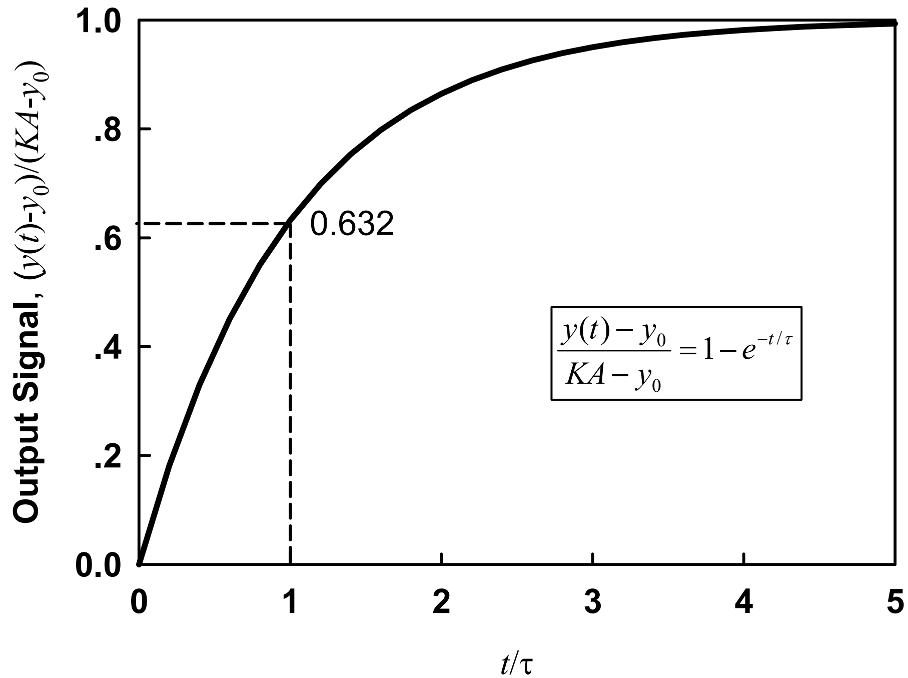
Applying the initial condition, we get $C = y_0 - KA$, thus gives

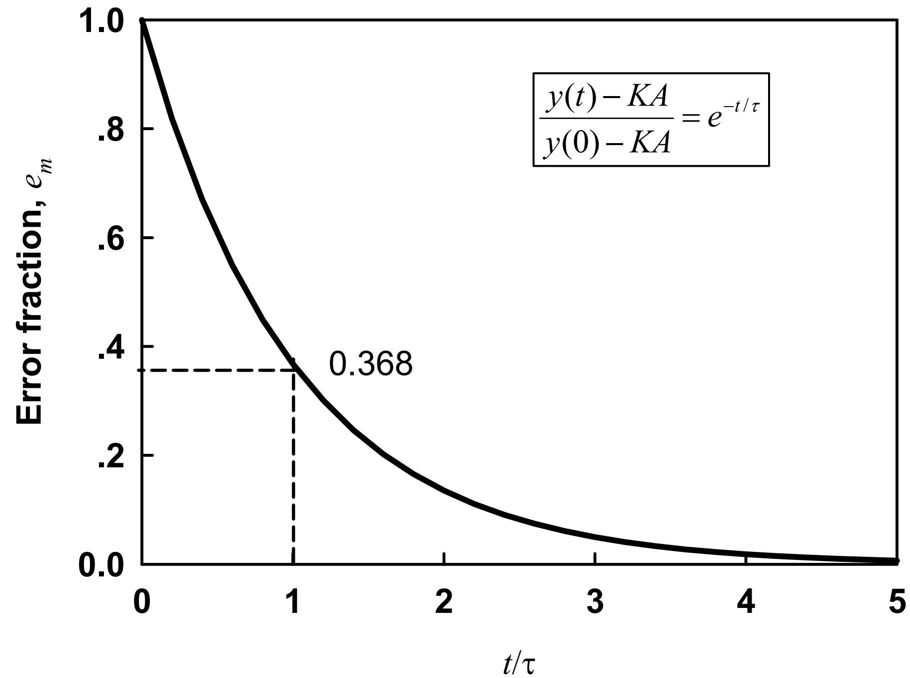
$$\boxed{y(t) = KA + (y_0 - KA)e^{-t/\tau}}$$

First-Order Systems: Step Response

Here, we define the term error fraction as

$$e_m(t) = \frac{y(t) - KA}{y_0 - KA} = \frac{y(t) - y(\infty)}{y(0) - y(\infty)} = e^{-t/\tau}$$



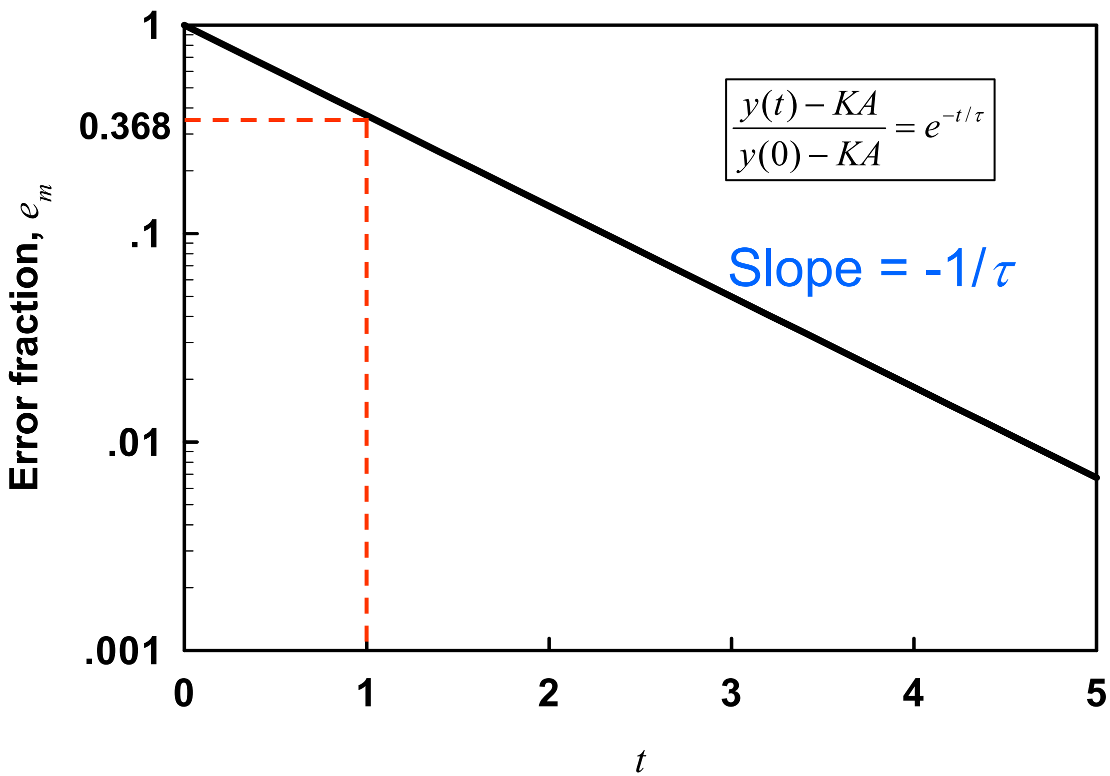


Non-dimensional step response of first-order instrument

Determination of Time constant

$$e_m = \frac{y(t) - KA}{y(0) - KA} = e^{-t/\tau}$$

$$\ln e_m = 2.3 \log e_m = -\frac{t}{\tau}$$



First-Order Systems: Ramp Response

Assume that at initial condition, both y and x = 0, at time = 0, the input quantity start to change at a constant rate $\dot{q}_{is}$ Thus, we have

$$x(t) = \begin{cases} 0 & t \leq 0 \\ \dot{q}_{is} t & t > 0 \end{cases}$$

Therefore

$$\tau \frac{dy(t)}{dt} + y(t) = K\dot{q}_{is} t U(t)$$

The complete solution:

$$y(t) = \underbrace{Ce^{-t/\tau}}_{\text{Transient response}} + \underbrace{K\dot{q}_{is}(t-\tau)}_{\text{Steady state response}}$$

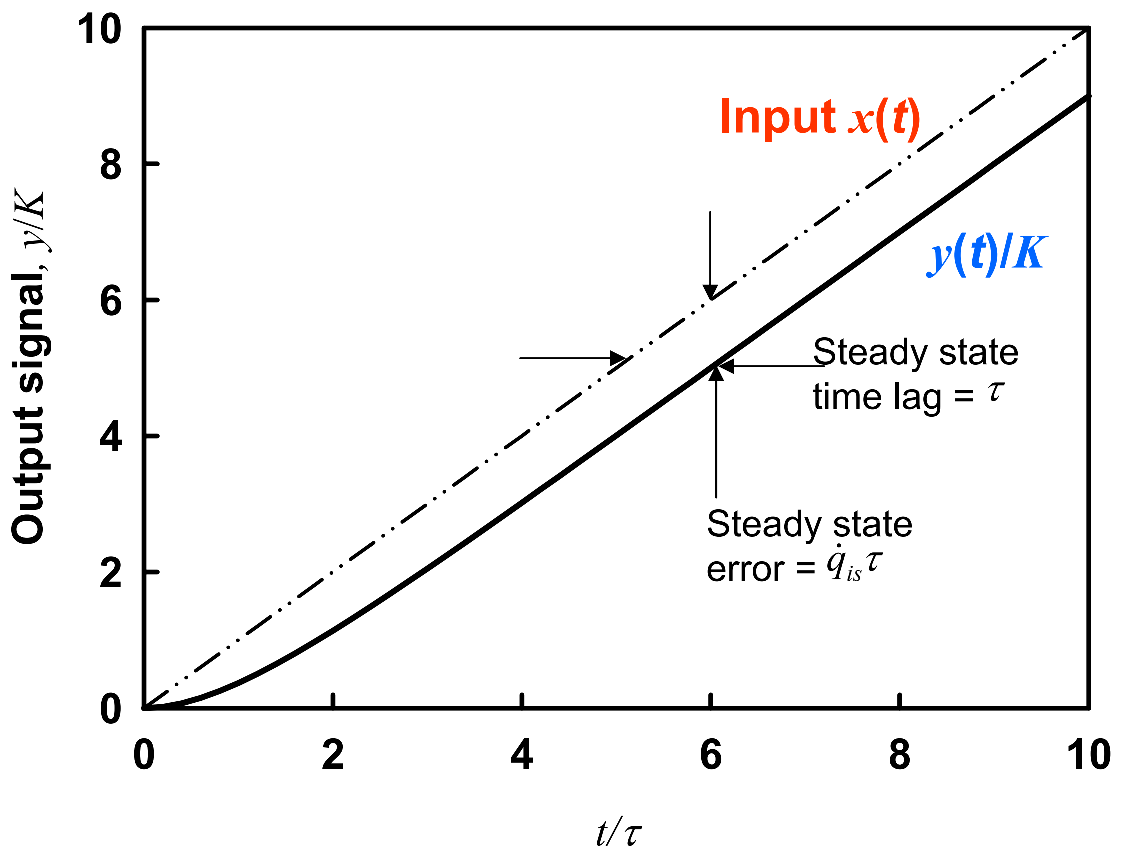
Applying the initial condition, gives

$$\boxed{y(t) = K\dot{q}_{is}(\tau e^{-t/\tau} + t - \tau)}$$

Measurement error

$$\boxed{e_m = x(t) - \frac{y(t)}{K} = \underbrace{-\dot{q}_{is}\tau e^{-t/\tau}}_{\text{Transient error}} + \underbrace{\dot{q}_{is}\tau}_{\text{Steady state error}}}$$

First-Order Systems: Ramp Response



Non-dimensional ramp response of first-order instrument

First-Order Systems: Frequency Response

From the response of first-order system to sinusoidal inputs, $x(t) = A\sin\omega t$ we have

$$\tau\frac{dy}{dt} + y = KA\sin\omega t \quad \Leftrightarrow \quad (\tau D + 1)y(t) = KA\sin\omega t$$

The complete solution:

$$y(t) = \underbrace{Ce^{-t/\tau}}_{\text{Transient response}} + \underbrace{\frac{KA}{\sqrt{1+(\omega\tau)^2}}\sin\left(\omega t - \tan^{-1}\omega\tau\right)}_{\text{Steady state response}}$$

Transient response

Steady state response = Frequency response

If we do interest in only steady state response of the system, we can write the equation in general form

$$y(t) = Ce^{-t/\tau} + B(\omega)\sin[\omega t + \phi(\omega)]$$

$$B(\omega) = \frac{KA}{\left[1+(\omega\tau)^2\right]^{1/2}}$$

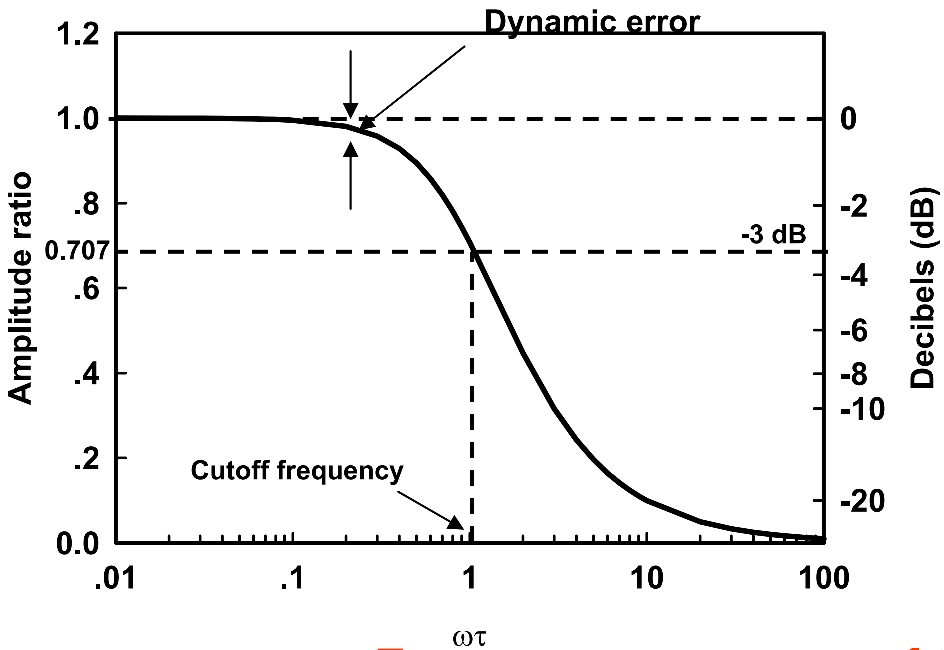
$$\phi(\omega) = -\tan^{-1}\omega\tau$$

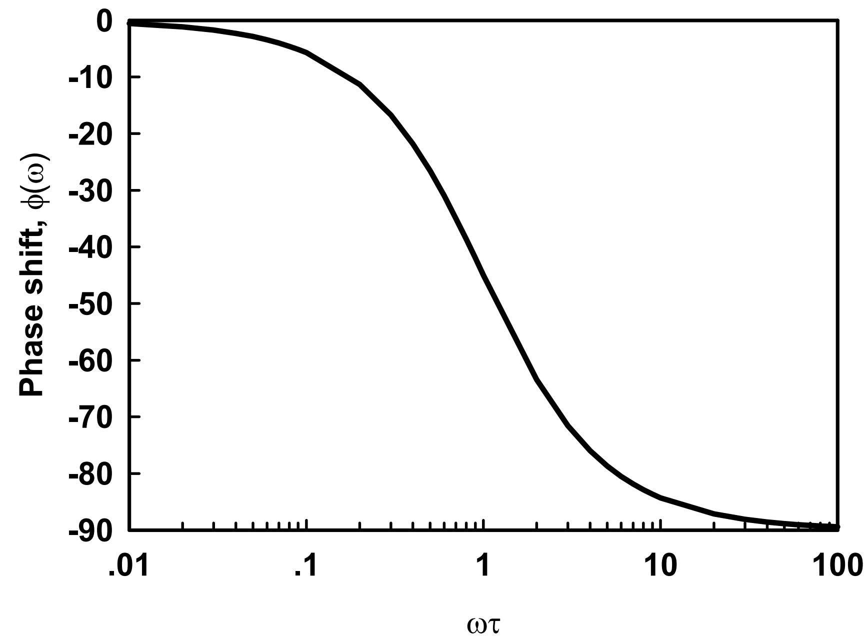
Where $B(\omega)$ = amplitude of the steady state response and $\phi(\omega)$ = phase shift

First-Order Systems: Frequency Response

$$M(\omega) = \frac{B}{KA} = \frac{1}{\left[1 + (\omega\tau)^2\right]^{1/2}}$$

The amplitude ratio $M(\omega) = \frac{1}{\sqrt{(\omega\tau)^2 + 1}}$ The phase angle is $\phi(\omega) = -\tan^{-1}(\omega\tau)$



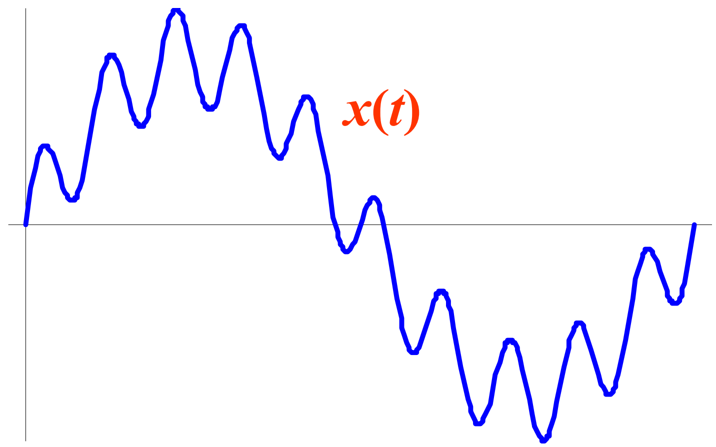


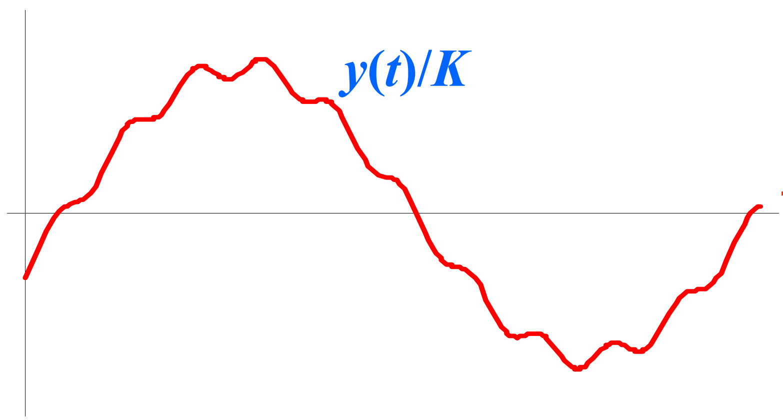
Frequency response of the first order system

Dynamic error, $\delta(\omega) = M(\omega) - 1$: a measure of an inability of a system to adequately reconstruct the amplitude of the input for a particular frequency

First-Order Systems: Frequency Response

Ex: Inadequate frequency response





Suppose we want to measure

$$x(t) = \sin 2t + 0.3 \sin 20t$$

With a first-order instrument whose τ is 0.2 s and static sensitivity K

Superposition concept:

For ω = 2 rad/s: $\frac{B}{A}(2\text{ rad/s}) = \frac{K}{\sqrt{0.16+1}} \angle -21.8^{\circ} = 0.93K \angle -21.8^{\circ}$

For ω = 20 rad/s: $\frac{B}{A}(20\text{ rad/s}) = \frac{K}{\sqrt{16+1}} \angle -76^{\circ} = 0.24K \angle -76^{\circ}$

Therefore, we can write $y(t)$ as

$$y(t) = (1)(0.93K)\sin(2t - 21.8^{\circ}) + (0.3)(0.24K)\sin(20t - 76^{\circ})$$

$$y(t) = 0.93K \sin(2t - 21.8^{\circ}) + 0.072K \sin(20t - 76^{\circ})$$

Second-Order Systems

In general, a second-order measurement system subjected to arbitrary input, $x(t)$

$$a_2 \frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t) \Rightarrow \left(\frac{D^2}{\omega_n^2} + \frac{2\zeta}{\omega_n} D + 1 \right) y(t) = Kx(t)$$

$$\frac{1}{\omega_n^2} \frac{d^2 y(t)}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dy(t)}{dt} + y(t) = Kx(t)$$

The essential parameters

$K = \frac{b_0}{a_0}$ = the static sensitivity

$\zeta = \frac{a_1}{2\sqrt{a_0 a_2}}$ = the damping ratio, dimensionless

$\omega_n = \sqrt{\frac{a_0}{a_2}}$ = the natural angular frequency

Second-Order Systems

Consider the characteristic equation

$$\frac{1}{\omega_n^2}D^2 + \frac{2\zeta}{\omega_n}D + 1 = 0$$

This quadratic equation has two roots:

$$S_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Depending on the value of ζ, three forms of complementary solutions are possible

Overdamped (ζ > 1):

$$y_{oc}(t) = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

Critically damped (ζ = 1):

$$y_{oc}(t) = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t}$$

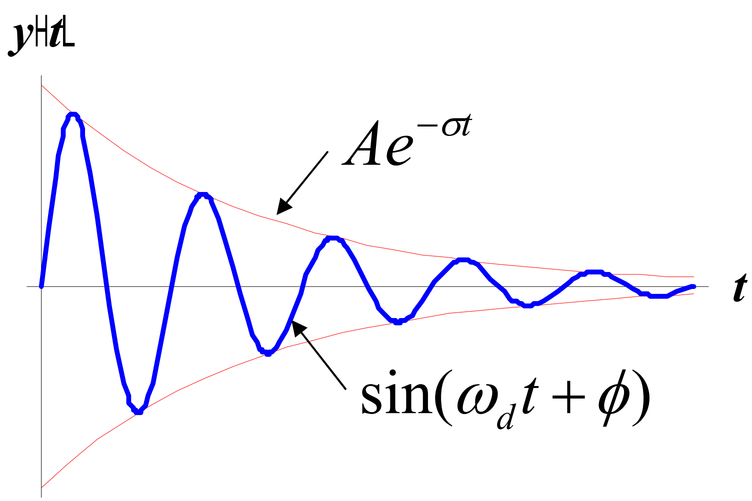
Underdamped (ζ< 1): :

$$y_{oc}(t) = C e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \Phi\right)$$

Second-Order Systems

Case I Underdamped (ζ< 1):

$$S_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$
$$= \sigma \pm j\omega_d$$

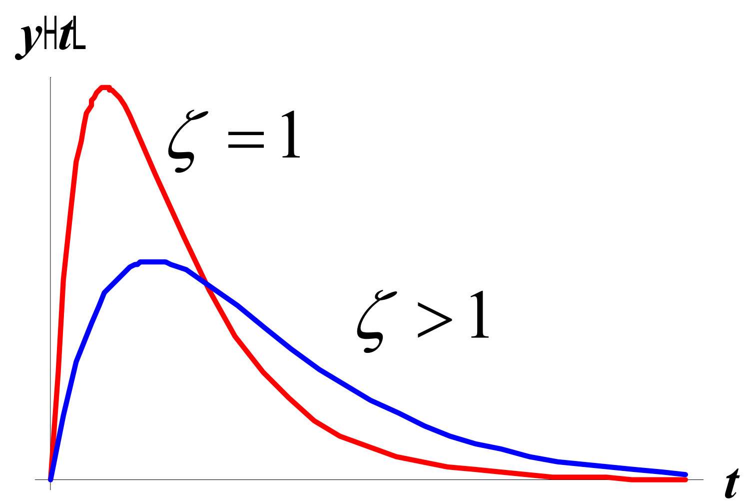


Case 2 Overdamped (ζ > 1):

$$S_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

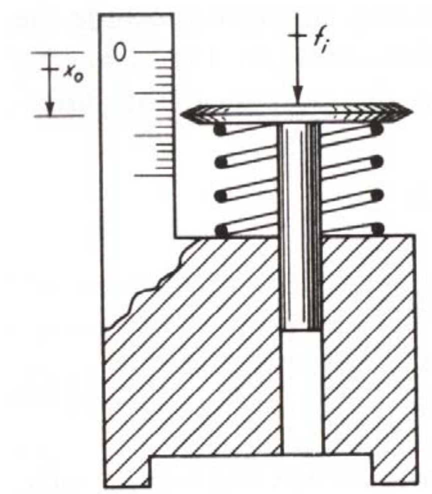
Case 3 Critically damped (ζ = 1):

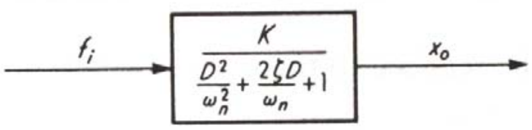
$$S_{1,2} = -\omega_n$$



Second-order Systems

Example: The force-measuring spring





consider a spring with spring constant K_s under applied force f_i and the total mass M. At start, the scale is adjusted so that x_o = 0 when f_i = 0;

$$\Sigma\text{forces}=(\text{mass})(\text{acceleration})$$

$$f_i - B\frac{dx_o}{dt} - K_s x_o = M\frac{d^2 x_o}{dt^2}$$

$$(MD^2 + BD + K_s)x_o = f_i$$

the second-order model:

$$K = \frac{1}{K_s} \qquad \text{m/N}$$

$$\omega_n = \sqrt{\frac{K_s}{M}} \qquad \text{rad/s}$$

$$\zeta = \frac{B}{2\sqrt{K_s M}}$$

Second-order Systems: Step Response

For a step input $x(t)$ $\quad \frac{1}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y=KAU(t) \Longleftrightarrow \left(\frac{D^2}{\omega_n^2}+\frac{2\zeta}{\omega_n}D+1\right)y(t)=KAU(t)$

With the initial conditions: y = 0 at t = 0+, dy/dt = 0 at t = 0+

The complete solution:

Overdamped (ζ > 1):

$$\frac{y(t)}{KA}=-\frac{\zeta+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}+\frac{\zeta-\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}+1$$

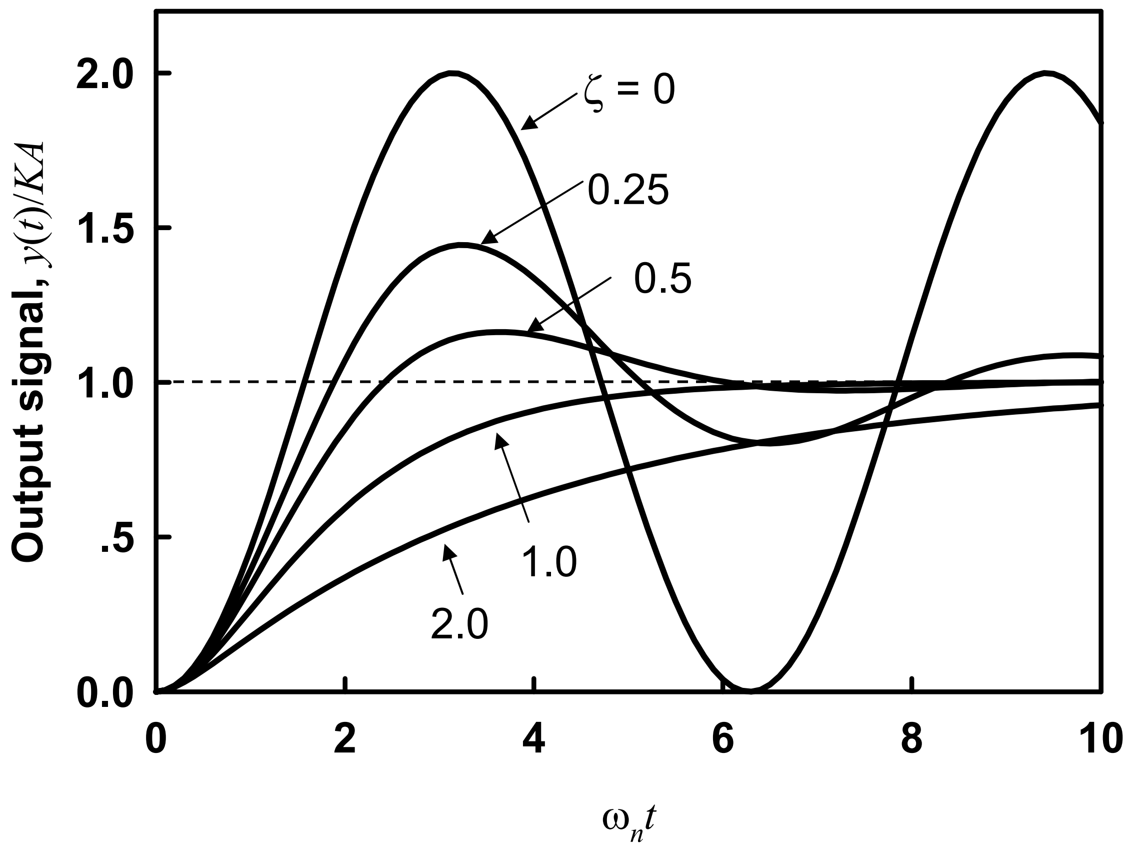
Critically damped (ζ = 1):

$$\frac{y(t)}{KA}=-(1+\omega_n t)e^{-\omega_n t}+1$$

Underdamped (ζ< 1): :

$$\frac{y(t)}{KA}=-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t+\phi\right)+1 \qquad \phi=\sin^{-1}\left(\sqrt{1-\zeta^2}\right)$$

Second-order Systems: Step Response



Ringing period $T_d = \frac{2\pi}{\omega_d}$

Ringing frequency: $\omega_d = \omega_n \sqrt{1-\zeta^2}$

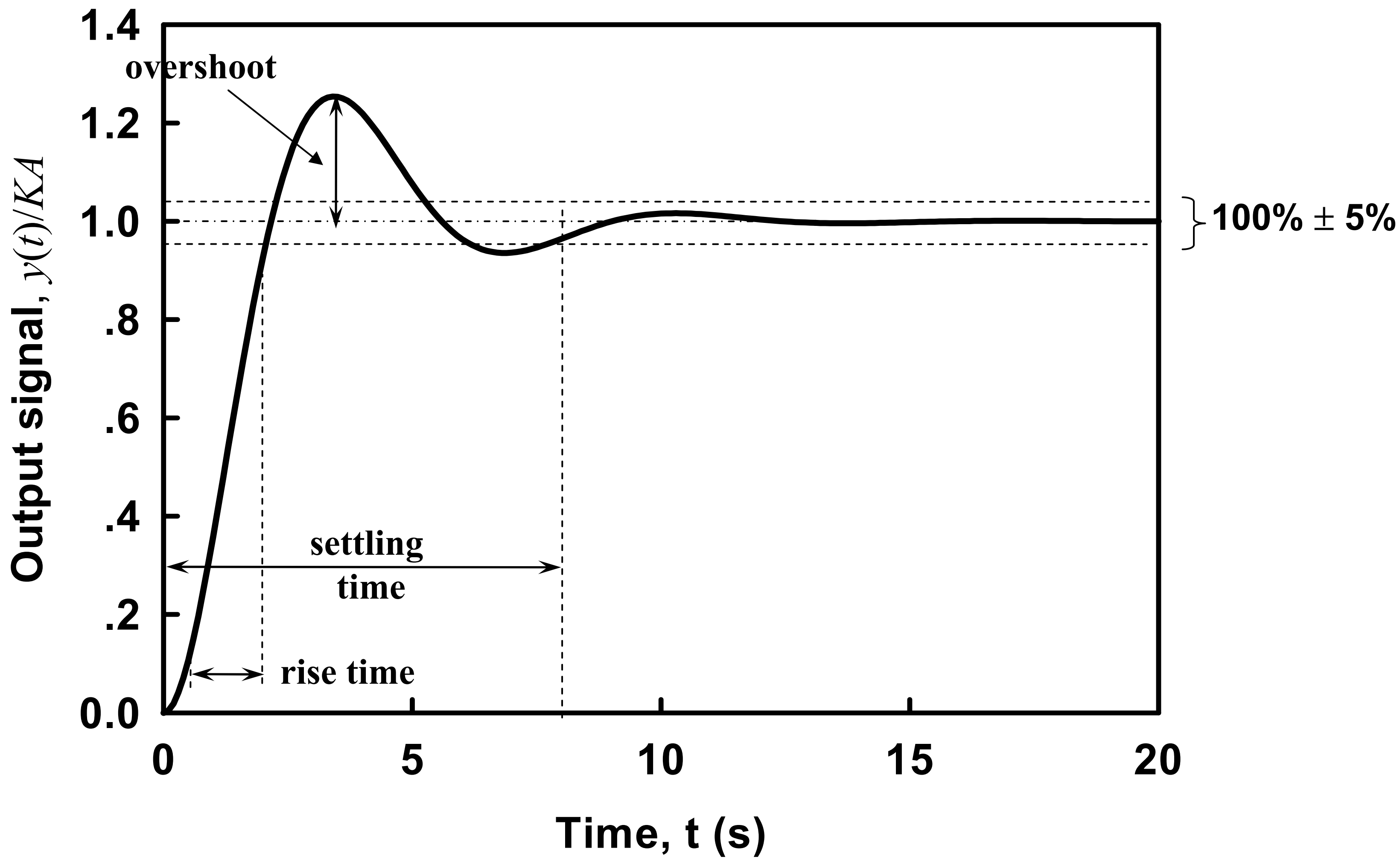
Rise time decreases ζ with but increases ringing

Optimum settling time can be obtained from $\zeta \sim 0.7$

Practical systems use $0.6 < \zeta < 0.8$

Non-dimensional step response of second-order instrument

Second-order Systems: Step Response



Typical response of the 2nd order system

Second-order Systems: Ramp Response

For a ramp input $x(t) = \dot{q}_{is} t U(t)$ $\quad \frac{1}{\omega_n^2}\frac{d^2 y}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy}{dt} + y = K\dot{q}_{is} t U(t)$

With the initial conditions: $y = dy/dt = 0$ at $t = 0+$

The possible solutions:

Overdamped:

$$\frac{y(t)}{K} = \dot{q}_{is} t - \frac{2\zeta\dot{q}_{is}}{\omega_n}\left(1 + \frac{2\zeta^2 - 1 - 2\zeta\sqrt{\zeta^2 - 1}}{4\zeta\sqrt{\zeta^2 - 1}} e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} + \frac{-2\zeta^2 + 1 - 2\zeta\sqrt{\zeta^2 - 1}}{4\zeta\sqrt{\zeta^2 - 1}} e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t}\right)$$

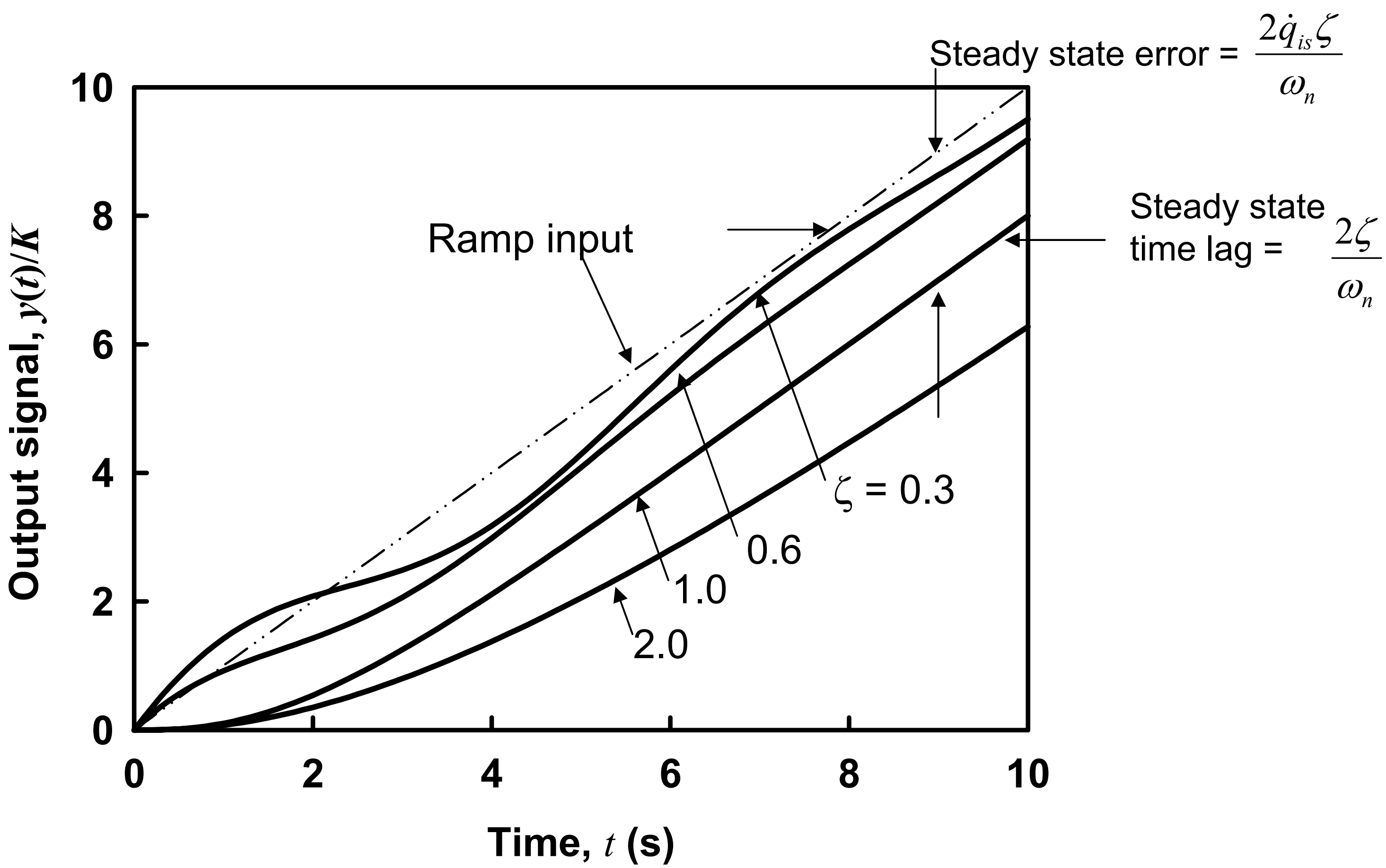
Critically damped:

$$\frac{y(t)}{K} = \dot{q}_{is} t - \frac{2\dot{q}_{is}}{\omega_n}\left[1 - (1 + \frac{\omega_n t}{1})e^{-\omega_n t}\right]$$

Underdamped:

$$\frac{y(t)}{K} = \dot{q}_{is} t - \frac{2\zeta\dot{q}_{is}}{\omega_n}\left[1 - \frac{e^{-\zeta\omega_n t}}{2\zeta\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi\right)\right] \qquad \phi = \tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{2\zeta^2 - 1}$$

Second-order Systems: Ramp Response



Typical ramp response of second-order instrument

Second-order Systems: Frequency Response

The response of a second-order to a sinusoidal input of the form $x(t) = A\sin\omega t$

$$y(t) = y_{ocf}(t) + \frac{KA}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2\right\}^{1/2}}\sin[\omega t + \phi(\omega)]$$

where $\phi(\omega) = -\tan^{-1}\dfrac{2\zeta}{\omega/\omega_n - \omega_n/\omega}$

The steady state response of a second-order to a sinusoidal input

$$y_{\text{steady}}(t) = B(\omega)\sin[\omega t + \phi(\omega)]$$

$$B(\omega) = \frac{KA}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2\right\}^{1/2}} \qquad \phi(\omega) = -\tan^{-1}\frac{2\zeta}{\omega/\omega_n - \omega_n/\omega}$$

Where $B(\omega)$ = amplitude of the steady state response and $\phi(\omega)$ = phase shift

$$M(\omega) = \frac{B}{KA} = \frac{1}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2\right\}^{1/2}}$$

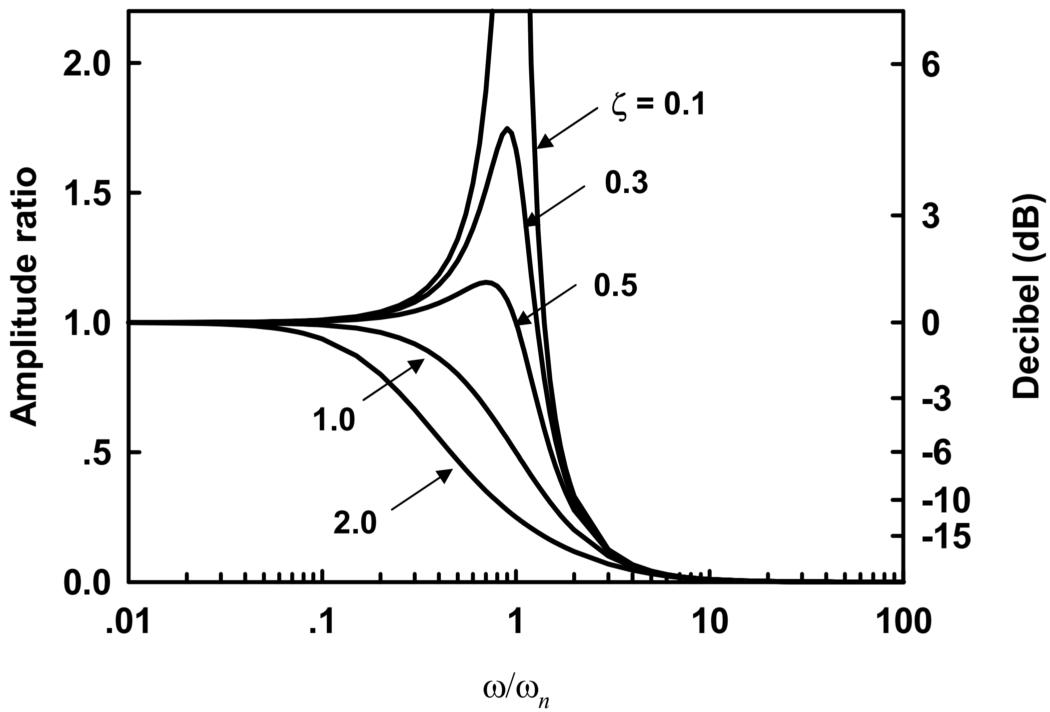
Second-order Systems: Frequency Response

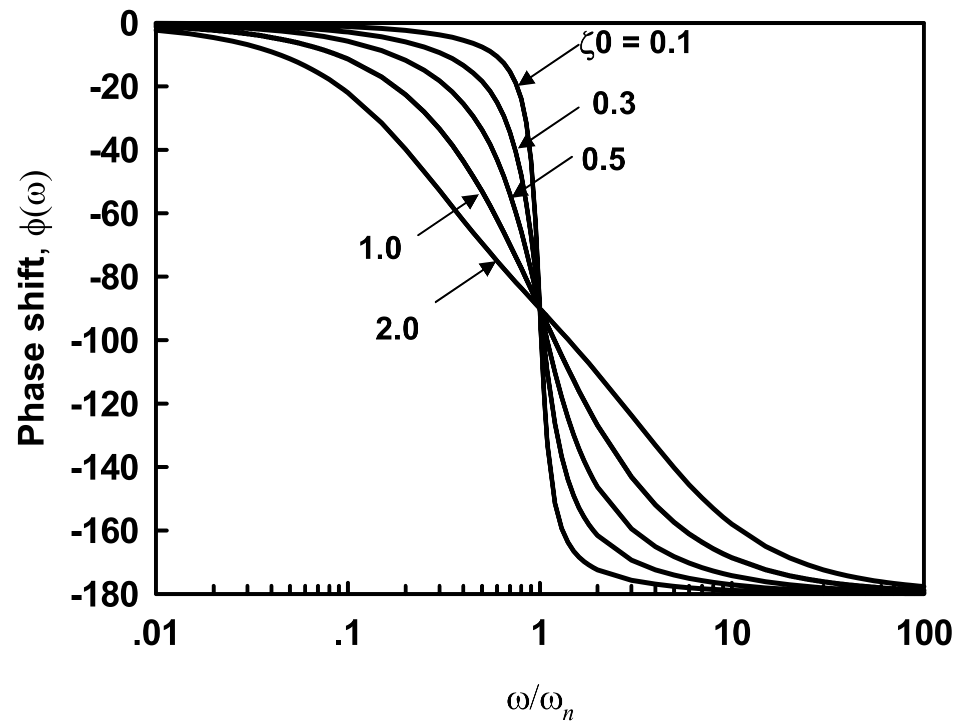
The amplitude ratio

$$M(\omega) = \frac{1}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2\right\}^{1/2}}$$

The phase angle

$$\phi(\omega) = -\tan^{-1}\frac{2\zeta}{\omega/\omega_n - \omega_n/\omega}$$





Magnitude and Phase plot of second-order Instrument

Second-order Systems

For overdamped (ζ >1) or critical damped (ζ = 1), there is neither overshoot nor steady-state dynamic error in the response.

In an underdameped system (ζ < 1) the steady-state dynamic error is zero, but the speed and overshoot in the transient are related.

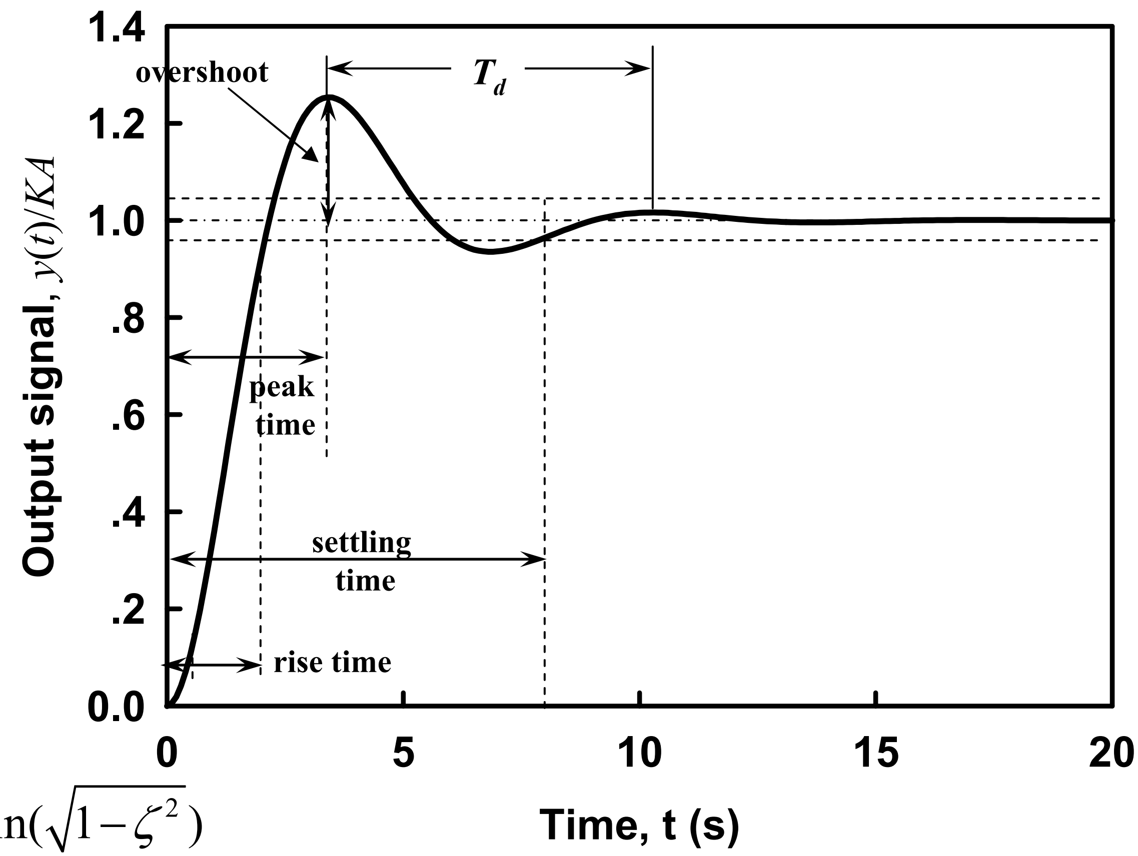
Rise time: $t_r = \dfrac{\arctan(-\omega_d / \delta)}{\omega_d}$

Maximum overshoot: $M_p = \exp\left(-\pi\zeta / \sqrt{1-\zeta^2}\right)$

Peak time: $t_p = \dfrac{\pi}{\omega_d}$

Resonance frequency: $\omega_r = \omega_n\sqrt{1-2\zeta^2}$

Resonance amplitude: $M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$



where $\delta = \zeta\omega_n, \omega_d = \omega_n\sqrt{1-\zeta^2}$, and $\phi = \arcsin(\sqrt{1-\zeta^2})$

Dynamic Characteristics

Speed of response: indicates how fast the sensor (measurement system) reacts to changes in the input variable. (Step input)

Rise time: the length of time it takes the output to reach 90% of full response when a step is applied to the input

Time constant: (1^{st} order system) the time for the output to change by 63.2% of its maximum possible change.

Settling time: the time it takes from the application of the input step until the output has settled within a specific band of the final value.

Transfer Function: a simple, concise and complete way of describing the sensor or system performance

$$H(s) = Y(s)/X(s)$$

where Y(s) and X(s) are the Laplace Transforms of the input and output respectively. Sometimes, the transfer function is displayed graphically as magnitude and phase plots VS frequency (Bode plot).

Dynamic Characteristics

Frequency Response describe how the ratio of output and input changes with the input frequency. (sinusoidal input)

Dynamic error, $\delta(\omega) = M(\omega) - 1$ a measure of the inability of a system or sensor to adequately reconstruct the amplitude of the input for a particular frequency

Bandwidth the frequency band over which $M(\omega) \geq 0.707$ (-3 dB in decibel unit)

Cutoff frequency: the frequency at which the system response has fallen to 0.707 (-3 dB) of the stable low frequency.

$$t_r \approx \frac{0.35}{f_c}$$

Dynamic Characteristics

Example: A first order instrument is to measure signals with frequency content up to 100 Hz with an inaccuracy of 5%. What is the maximum allowable time constant? What will be the phase shift at 50 and 100 Hz?

Solution: Define $M(\omega) = \frac{1}{\sqrt{\omega^2\tau^2+1}}$

$$\text{Dynamic error} = \left(M(\omega)-1\right)\times 100\% = \left(\frac{1}{\sqrt{\omega^2\tau^2+1}}-1\right)\times 100\%$$

From the condition |Dynamic error| < 5%, it implies that $0.95 \le \frac{1}{\sqrt{\omega^2\tau^2+1}} \le 1.05$

But for the first order system, the term $1/\sqrt{\omega^2\tau^2+1}$ can not be greater than 1 so that the constrain becomes

$$0.95 \le \frac{1}{\sqrt{\omega^2\tau^2+1}} \le 1$$

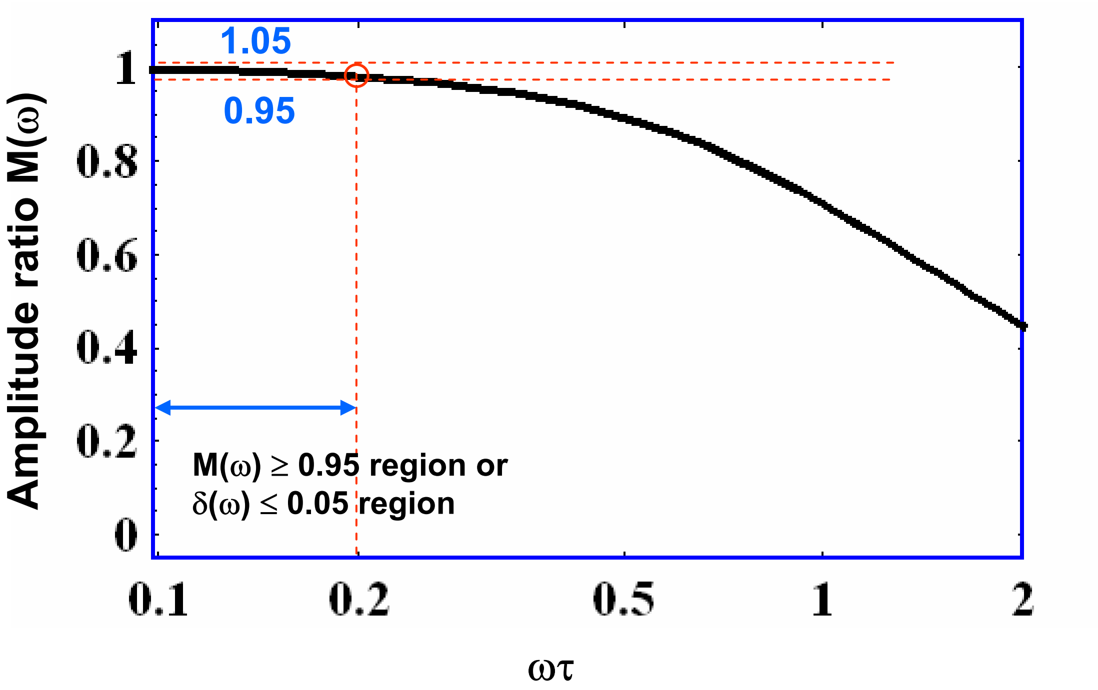
Solve this inequality give the range $0 \le \omega\tau \le 0.33$

The largest allowable time constant for the input frequency 100 Hz is $\tau = \frac{0.33}{2\pi 100\ \text{Hz}} = 0.52\ \text{ms}$

The phase shift at 50 and 100 Hz can be found from $\phi = -\arctan \omega\tau$

This give ϕ = -9.33^o and = -18.19^o at 50 and 100 Hz respectively

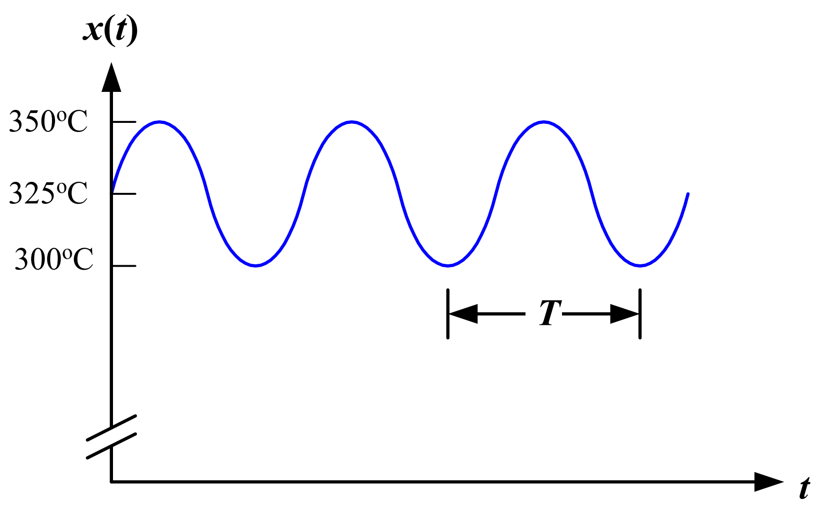
Dynamic Characteristics

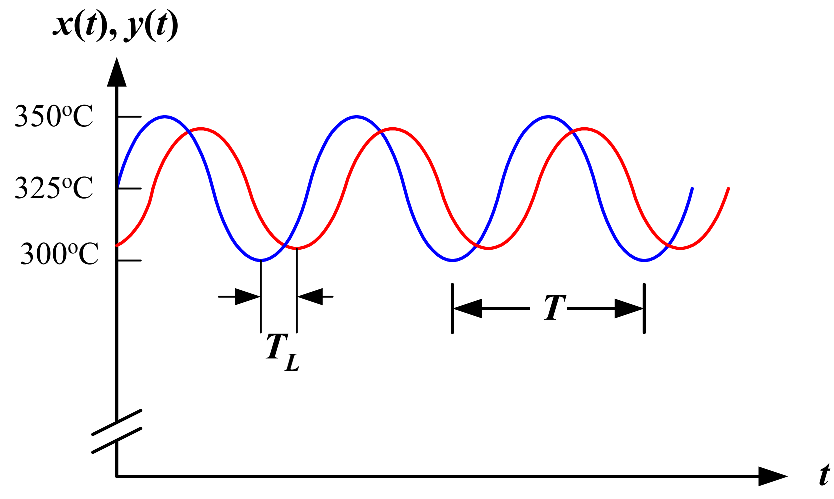


Dynamic Characteristics

Example: A temperature measuring system, with a time constant 2 s, is used to measured temperature of a heating medium, which changes sinusoidal between 350 and 300°C with a periodic of 20 s. find the maximum and minimum values of temperature, as indicated by the measuring system and the time lag between the output and input signals

Solution: $y(t) = 325 + 21.2\sin(\frac{2\pi}{60}t - 0.56)^{\circ}\mathrm{C}$





Dynamic Characteristics

Example: The approximate time constant of a thermometer is determined by immersing it in a bath and noting the time it takes to reach 63% of the final reading. If the result is 28 s, determine the delay when measuring the temperature of a bath that is periodically changing 2 times per minute.

$$t_d = 6.7\text{s}$$

Dynamic Characteristics

Example: A pressure transducer has a natural frequency of 30 rad/s, damping ratio of 0.1 and static sensitivity of 1.0 μV/Pa. A step pressure input of 8x10^5 N/m^2 is applied. Determine the output of a transducer.

Solution:

$$y(t) = 0.8[1 - e^{-3t} \sin(29.85t + 1.47)] \quad \text{V}$$

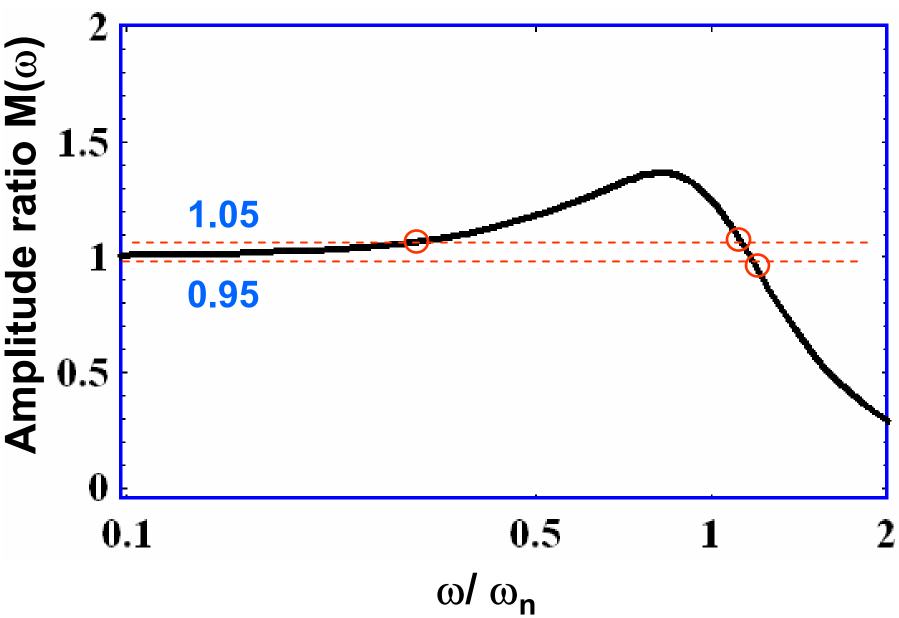
Example: A second order instrument is subjected to a sinusoidal input. Undamped natural frequency is 3 Hz and damping ratio is 0.5. Calculate the amplitude ratio and phase angle for an input frequency of 2 Hz.

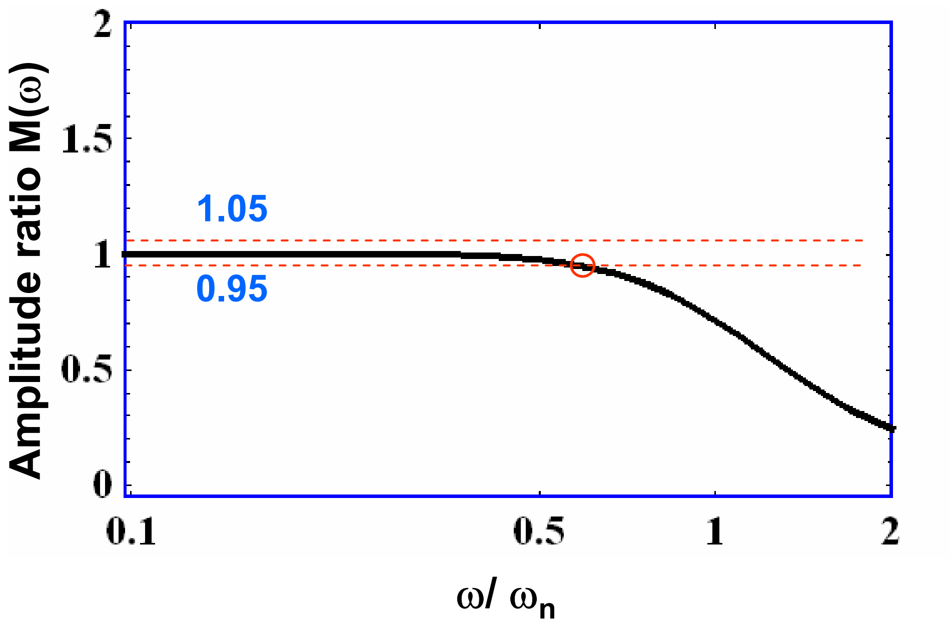
Solution: Amplitude ratio $y(t)/x(t)$ = 1.152 and phase shifts –50.2^o.

Dynamic Characteristics

Example: An Accelerometer is to selected to measure a time-dependent motion. In particular, input signal frequencies below 100 Hz are of prime interest. Select a set of acceptable parameter specifications for the instrument, assuming a dynamic error of ±5% and damping ratio ζ =0.7

Solution: $\omega_n \geq 1047$ rad/s





Response of a General Form of System to a Periodic Input

The steady state response of any linear system to the complex periodic signal can be determined using the frequency response technique and principle of superposition.

Let $x(t)$

$$x(t) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos n\omega_0 t + B_n \sin n\omega_0 t \right)$$

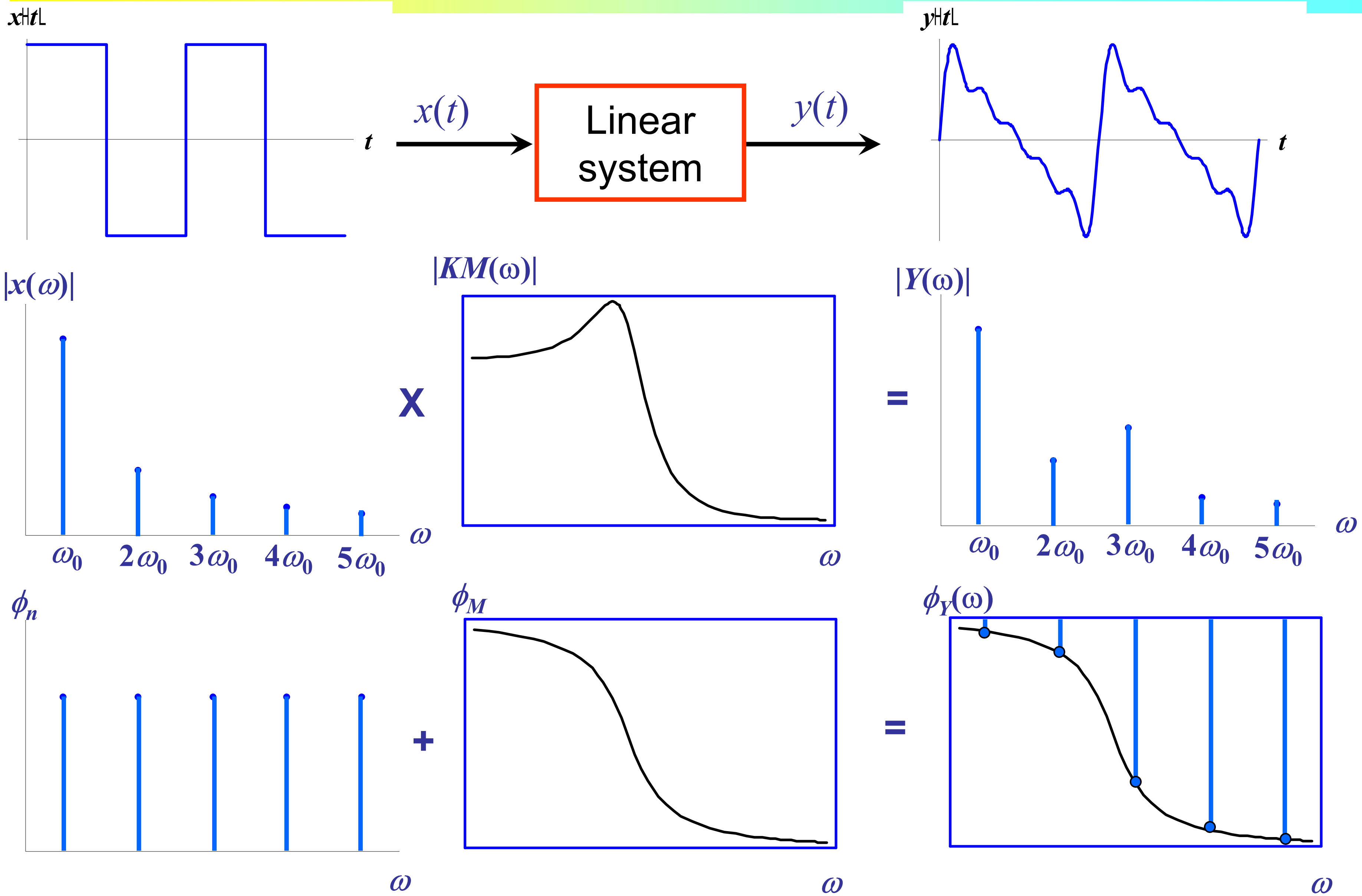
The frequency response of the measurement system

$$y(t) = KA_0 + \sum_{n=1}^{\infty} \left(\sqrt{A_n^2 + B_n^2} \right) KM(n\omega_0) \sin\left(n\omega_0 t + \phi_M(n\omega_0) + \phi_n(n\omega_0) \right)$$

Where $KM(\omega)$ = Magnitude of the frequency response of the measurement system and

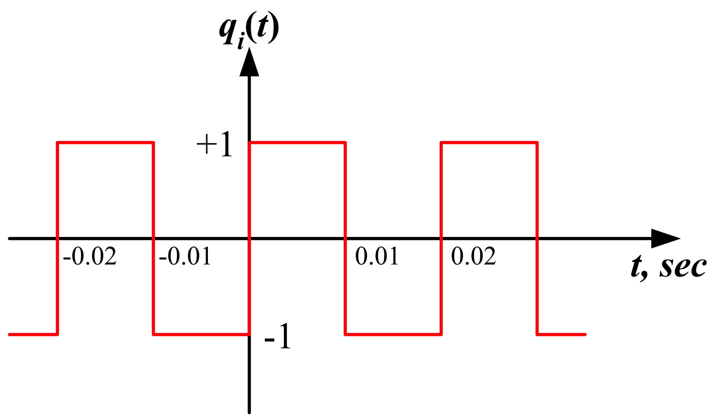
$\phi_M(\omega)$ = Phase shift = $\tan^{-1}(A_n/B_n)$

Response of a General Form of System of a Periodic Input



Response of a General Form of System to a Periodic Input

Example: If $q_i(t)$ as shown in Figure below is the input to a first-order system with a sensitivity of 1 and a time constant of 0.001 s, find $Q_o(i\omega)$ and $q_o(t)$ for the periodic steady state.



$$Q_o(i\omega) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\frac{1}{\sqrt{(n\omega_o\tau)^2+1}}\angle\phi_n\right]$$ Where n = odd number

$$q_o(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\frac{1}{\sqrt{(n\omega_o\tau)^2+1}}\sin(n\omega_o t+\phi_n)\right]$$ and $\phi_n = -\arctan(n\omega_o\tau)$