Factorizable Dependence

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Abstract

Random variables X and Y are *completely dependent* if one is completely determined by the other almost surely. In terms of Markov operators, complete dependence corresponds to left or right invertible Markov operators. We are interested in much more general *factorizable dependence* corresponding to a composition of left and right invertible Markov operators. Some results in a large subclass of singular Markov operators will be discussed culminating in an "algorithm" to verify whether such a Markov operator gives factorizable dependence.

Complete Dependence & Implicit Dependence

Definition

Random variables X and Y are *completely dependent* if

$$\exists \text{ a function } \alpha,\ Y \overset{\text{a.s.}}{=} \alpha(X) \quad \text{or} \quad \exists \text{ a function } \beta, X \overset{\text{a.s.}}{=} \beta(Y).$$

If $Y \overset{\text{a.s.}}{=} \alpha(X)$ and X, Y are uniformly distributed on $[0,1]$ $(X, Y \sim U[0,1])$ then α must be measure-preserving on $[0,1]$ (i.e. $\lambda\left(\alpha^{-1}(E)\right) = \lambda(E)$ for all $E \subseteq [0,1]$).

Definition

X and Y are *implicitly dependent* if $\exists \alpha, \beta,\ \alpha(X) \overset{\text{a.s.}}{=} \beta(Y)$.

Conditional expectations ⇒ Markov operators

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Definition

For random variables $X, Y \sim U[0,1]$, we define for $f \in L^1([0,1])$

$$(T_{X,Y} f)(x) = E(f \circ Y | X = x).$$

In particular, for $S \subseteq [0,1]$, $(T_{X,Y} \mathbb{1}_S)(x) = P(Y \in S | X = x)$.

Examples:

- $T_{X,X} \mathbb{1}_S(x) = \mathbb{1}_S(x)$.
- $T_{X,1-X} \mathbb{1}_S(x) = \mathbb{1}_{1-S}(x) = \mathbb{1}_S(1-x)$.
- If X and Y are independent, then $T_{X,Y} \mathbb{1}_S(x) = \lambda(S)$.

[1] $\mathbb{1}_S \equiv$ the characteristic function on S and $\lambda \equiv$ Lebesgue measure on $\mathbb{R}$

Markov operators

2

Definition

A linear operator $T\colon L^1(\mathbb{I}) \to L^1(\mathbb{I})$ is called a *Markov operator* if:

1. $f \geq 0$ implies $Tf \geq 0$ for all $f \in L^1(\mathbb{I})$,
2. $T\mathbb{1}_{\mathbb{I}} = \mathbb{1}_{\mathbb{I}}$, and
3. $\int_{\mathbb{I}} Tf \, \mathrm{d}\lambda = \int_{\mathbb{I}} f \, \mathrm{d}\lambda$ for all $f \in L^1(\mathbb{I})$.

Fact: For every r.v.'s X and Y, $T_{X,Y}$ is a Markov operator and every Markov operator arises this way.

[2] $\mathbb{I} = [0, 1]$.

Invertibility

More facts:

- The class $\mathscr{T}$ of Markov operators is closed under composition.
- The identity operator $Mf = f$ is the identity in $(\mathscr{T}, \circ)$.

Definition

$T \in \mathscr{T}$ is *left (right) invertible* if there exists $T' \in \mathscr{T}$ such that $T' \circ T = M$ $(T \circ T' = M)$. In fact, $T' = T^*$.

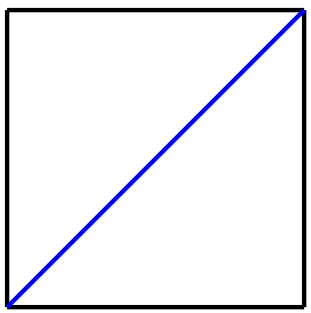
Example: For any measure-preserving function φ on $\mathbb{I}$, $T_\varphi f = f \circ \varphi$ defines a left invertible Markov operator whose left inverse is the adjoint of T_φ mapping $f \circ \varphi$ to f.

Invertibility = Complete dependence

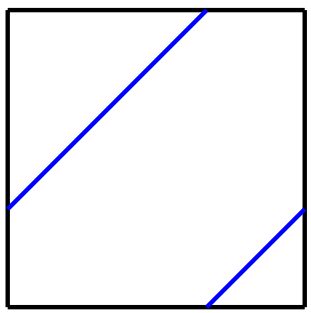
[Darsow, Nguyen and Olsen 1992]:

- $T_{X,Y}$ is left invertible if and only if $Y \overset{\text{a.s.}}{=} \alpha(X)$ for some α.
- $T_{X,Y}$ is right invertible if and only if $X \overset{\text{a.s.}}{=} \beta(Y)$ for some β.
- $T_{X,Y}$ is invertible if and only if $Y \overset{\text{a.s.}}{=} \alpha(X)$ for some 1-1 α.

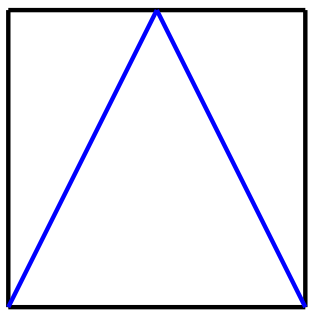
Examples:



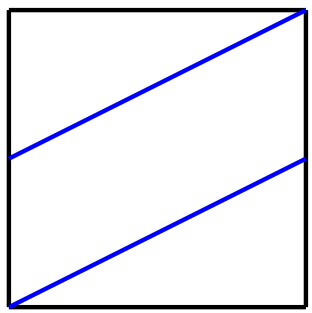
Identity



Invertible



Left invertible



Right invertible

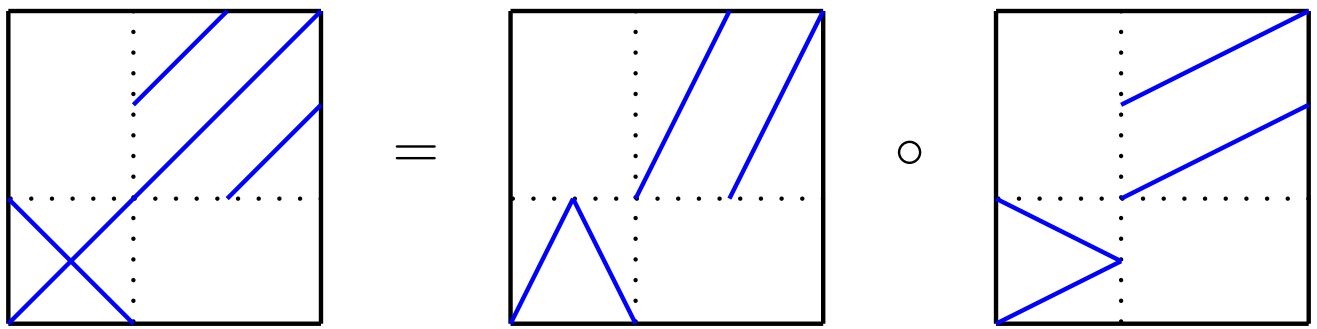
Idempotent Markov operators

Definition

A Markov operator T is *idempotent* if $T \circ T = T$.

[Darsow and Olsen 2010]: A Markov operator T is non-atomic & idempotent if and only if $T = L \circ L^*$ for some left invertible Markov operator L.

Example:



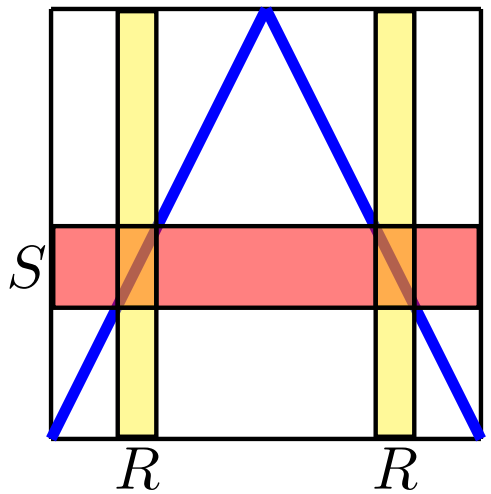
Associated σ-algebras

For a Markov operator T, consider sets $S, R \in \mathscr{B}$ such that

$$T\mathbb{1}_S = \mathbb{1}_R.$$

Define two associated *σ-algebras* by

$$\sigma_T \equiv \{\text{all such } S\text{'s}\} \qquad \text{and} \qquad \sigma_T^* \equiv \{\text{all such } R\text{'s}\}\,.$$



Then $\sigma_T = \mathscr{B}$ & $\sigma_T^* = \{R \in \mathscr{B} \colon x \in R \text{ iff } 1 - x \in R\}$.

Some Properties of σ_C & σ_C^*

- T maps σ_T-measurable functions to σ_T^*-measurable functions.
- $\sigma_T = \mathscr{B}$ if and only if T is left invertible.
- Moreover, if $T = T_\varphi$ (left invertible) then σ_T^* "=" $\varphi^{-1}(\mathscr{B})$.
- $\sigma_T^* = \mathscr{B}$ if and only if T is right invertible.
- If $T = T_\varphi^*$ (right invertible) then σ_T "=" $\varphi^{-1}(\mathscr{B})$.

Non-atomic Markov operators

Definition

Consider a sub-σ-algebra $\mathscr{S} \subseteq \mathscr{B}$ in $([0,1], \mathscr{B}, \lambda)$.

- $S \in \mathscr{S}$ is said to be an *atom* if $\lambda(S) > 0$ and there is no subset S' of S in $\mathscr{S}$ with $0 < \lambda(S') < \lambda(S)$.
- $\mathscr{S}$ is called *non-atomic* if there are no atoms in $\mathscr{S}$.

A Markov operator T is *non-atomic* if σ_T or σ_T^* is non-atomic.

Non-Examples: If X and Y are independent then $T = T_{X,Y}$ is **not** non-atomic. In fact, $T\mathbb{1}_S(x) = \lambda(S)$ implies that $\sigma_T = \sigma_T^* = \{S \in \mathscr{B} \colon \lambda(S) = 0 \text{ or } 1\}$ which has an atom.

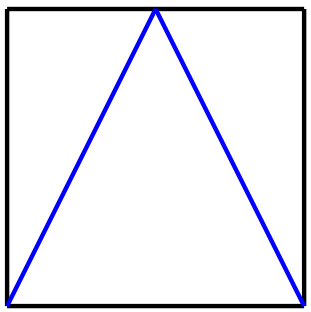
Properties of non-atomic Markov operators

- [S. 2017] Every non-atomic Markov operator must be singular, i.e. the support of the induced doubly stochastic measure

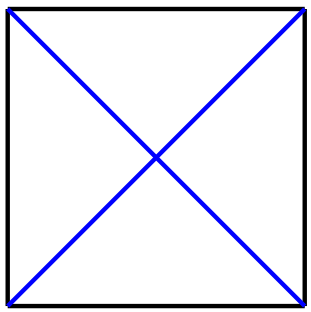
$$\mu_T(A \times B) = \int_A T\mathbb{1}_B \, d\lambda$$

has Lebesgue measure zero.

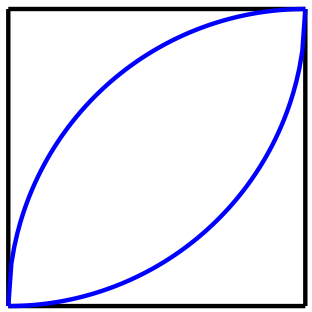
- The class of non-atomic Markov operators includes:



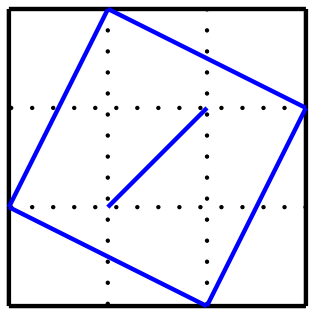
all complete dependence MO's,



all non-atomic idempotent MO's,



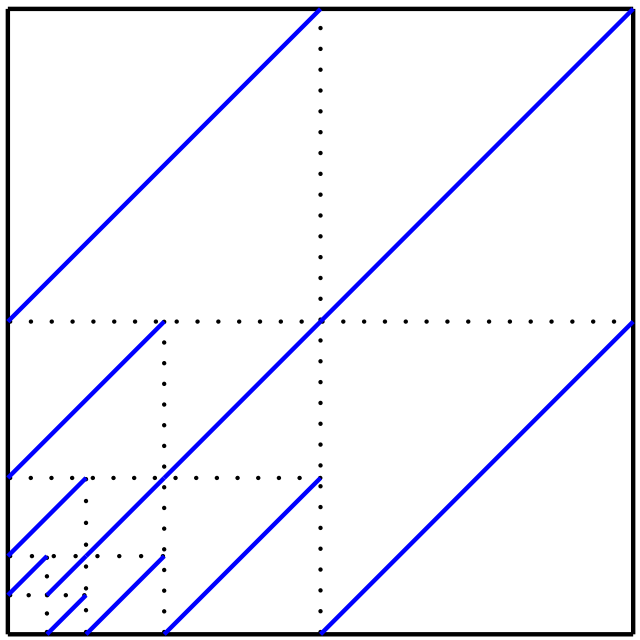
all hairpin MO's,



& more.

First problem

Conjecture 1: $\exists$ a singular MO that is not non-atomic, e.g.



Partial factorization of non-atomic M.O.'s

Theorem (S. 2017)

Let T be a Markov operator. TFAE.

1. *T is non-atomic.*
2. *$\exists$ measure-preserving functions f and g on $[0,1]$ such that $f^{-1}(\mathscr{B}) \subseteq \sigma_T^*$, $g^{-1}(\mathscr{B}) \subseteq \sigma_T$ and*

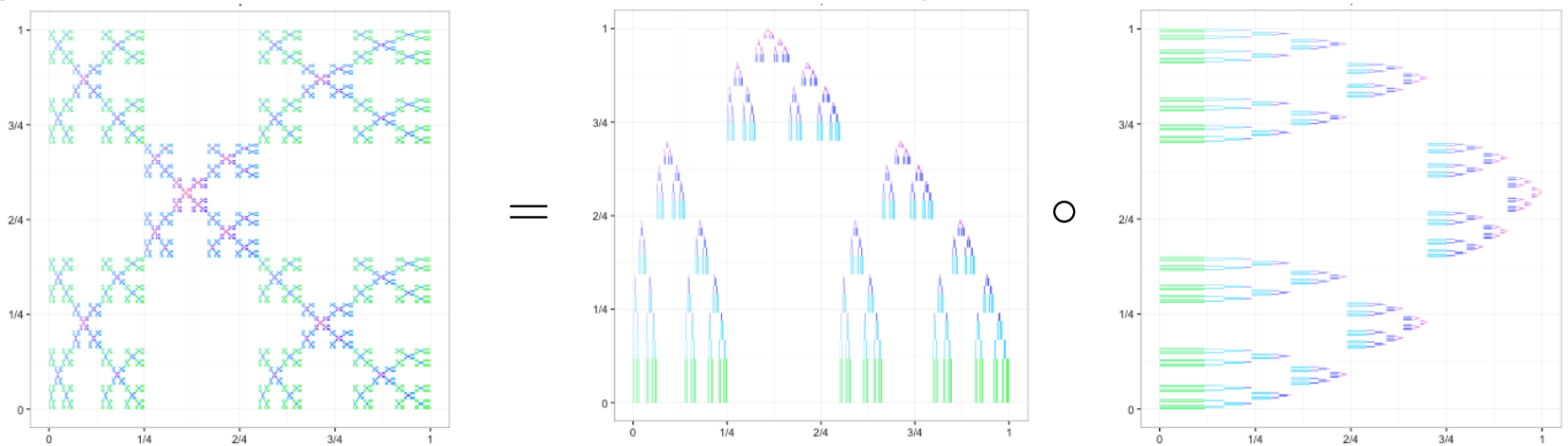
$$T = T_{U,f(U)} \circ T_{g(U),U} \quad \text{on } L^1([0,1], g^{-1}(\mathscr{B})).$$

Here, $U \sim U[0,1]$.

MOs with fractal support

Fact: For each $s \in (1, 2)$, there exists a MO whose support is a fractal set with dimension s.

Some of them can be factored as $L \circ R$:

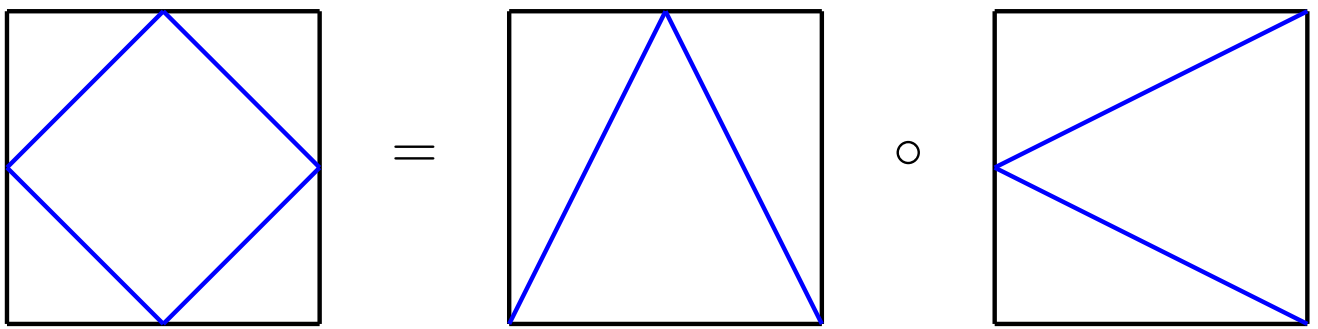


Factorizable Markov operators

Definition

A (non-atomic) Markov operator T is said to be *factorizable* if there exist measure-preserving functions f and g such that $T = T_{U,f(U)} \circ T_{g(U),U}$ on $L^1([0,1], \mathscr{B})$.

Example:

- Clearly, every non-atomic idempotent MO is factorizable.
- 

σ-algebras of factorizable Markov operators

[Printechapat and S.]: Let f, g be measure-preserving functions and $T = T_{U,f(U)} \circ T_{g(U),U}$.

- $\sigma_T = \sigma_{T_{g(U),U}}$ and $\sigma_T^* = \sigma^*_{T_{U,f(U)}}$.
- $T^* \circ T = T_{U,g(U)} \circ T_{g(U),U}$ and $T \circ T^* = T_{U,f(U)} \circ T_{f(U),U}$.
- For any MO T and 1-1 m-p function φ, $\sigma_{T_\varphi \circ T} = \sigma_T$ and $\sigma^*_{T_\varphi \circ T} = \varphi^{-1}(\sigma_T^*)$ ess.

Sharper partial factorization

[Darsow and Olsen 2010]: For every non-atomic σ-algebra $\mathscr{S}$, there exists a measure-preserving functions f such that $f^{-1}(\mathscr{B}) = \mathscr{S}$ ess., i.e. $\{\mathbb{1}_E : E \in f^{-1}(\mathscr{B})\} = \{\mathbb{1}_E : E \in \mathscr{S}\}$

Theorem (Printechapat and S.)

Let T be a non-atomic MO such that $f^{-1}(\mathscr{B}) = \sigma_T^*$ ess. and $g^{-1}(\mathscr{B}) = \sigma_T$ ess. Then

1. $S := T_{f(U),U} \circ T \circ T_{U,g(U)}$ *is invertible.*
2. $T = T_{U,f(U)} \circ S \circ T_{g(U),U}$ *on* $L^1([0,1], \sigma_T)$.

From non-atomic to factorizable

A procedure to construct (the) isoalgebra factorizable MO T' of a given non-atomic MO T:

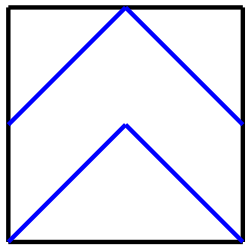
1. Choose m-p functions f, g such that $f^{-1}(\mathscr{B}) = \sigma_T^*$ ess. and $g^{-1}(\mathscr{B}) = \sigma_T$ ess.
2. Compute $S \equiv T_{f(U),U} \circ T \circ T_{U,g(U)}$.
3. Then $T' = T_{U,f(U)} \circ S \circ T_{g(U),U}$ has the same σ-algebras as T.

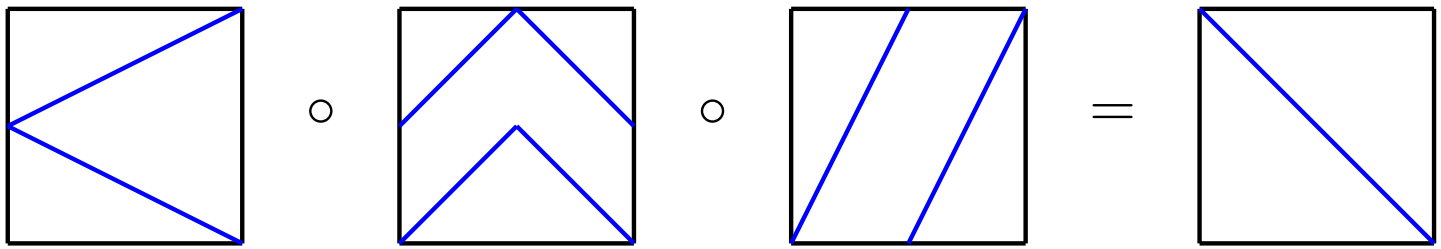
In fact, T' is independent of the choice of f, g.

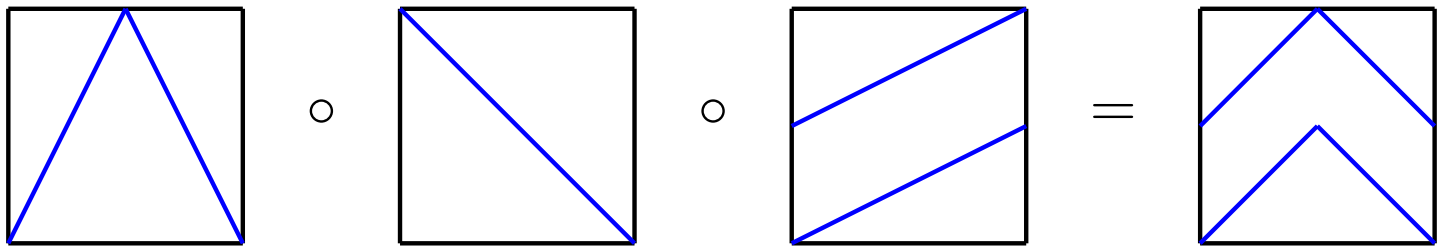
Factorizability procedure/test

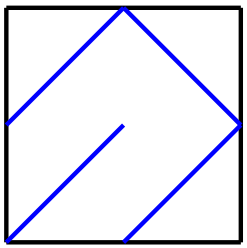
Let T be a non-atomic Markov operator.

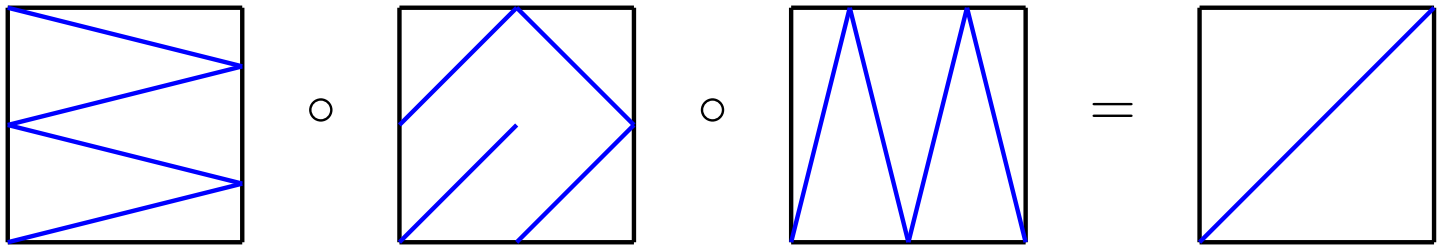
T is factorizable if and only if $T' = T$.

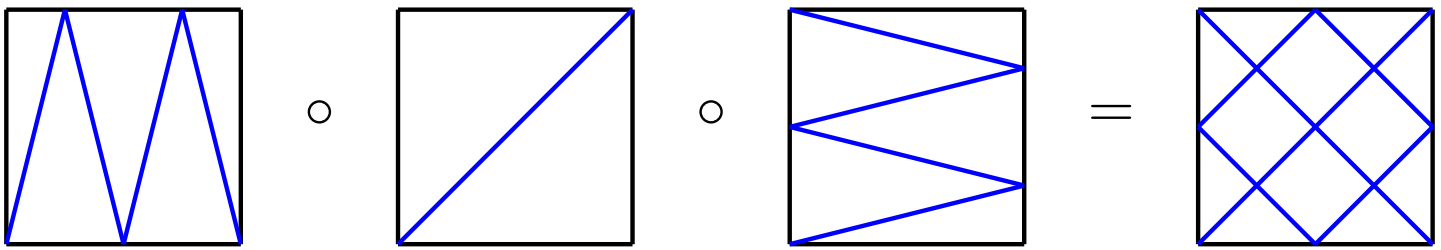
Example: $T =$  with uniform mass distribution. Then

$S =$ 

$T' =$ 

Example: $T =$  with uniform mass distribution. Then

$S =$ 

$T' =$ 

Second problem

Factorizable dependence vs. Implicit dependence:

Note: If $\alpha(X) \overset{\text{a.s.}}{=} \beta(Y)$ and

if X, Y are conditionally independent given $\alpha(X)$ then the

implicit dependence MO

$$T_{X,Y} = T_{X,\alpha(X)} \circ T_{\beta(Y),Y}.$$

Can every implicit dependence MO be factorized?

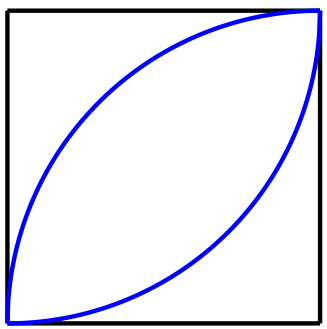
Conjecture 2: {implicit dependence MO's} = {factorizable MO's}

Hairpin Markov operators

Definition

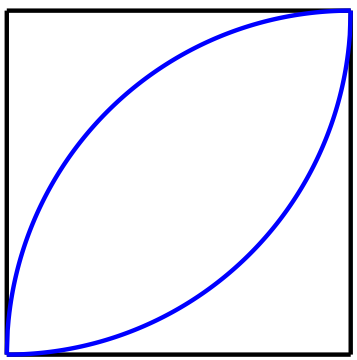
A Markov operator T is called a *hairpin MO* if its induced doubly stochastic measure μ_T is supported on the union of the graphs of h and h^{-1} where h is an increasing homeomorphism of $[0,1]$ onto itself such that $h(x) < x$ for $0 < x < 1$.

Example: If $h(x) = 1 - \sqrt{1-x^2}$

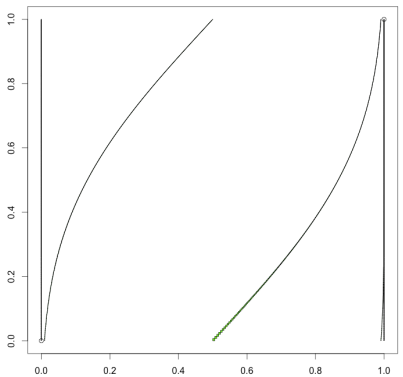
then $T =$  is not factorizable.

Goal: Most hairpin Markov operators are not factorizable.

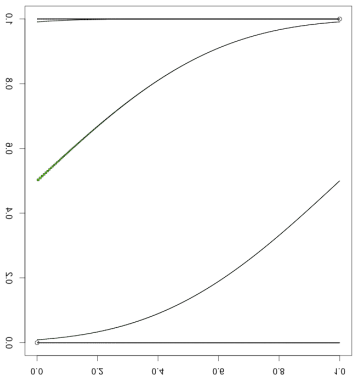
Hairpin Markov operators



$\neq$

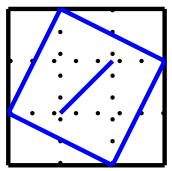


$\circ$



Third problem

Facts: Denote $\mathscr{T}$ = the convex set of Markov operators.

- All complete dependence Markov operators are extreme points in $\mathscr{T}$.
- All hairpin Markov operators are extreme points in $\mathscr{T}$.
-  is also an extreme point in $\mathscr{T}$.

Conjecture 3: Every extreme point in $\mathscr{T}$ is non-atomic.

Continuing research

1. $\{$non-atomic Markov operators$\} \subsetneq \{$singular Markov operators$\}$?
2. $\{$implicit dependence Markov operators$\} = \{$factorizable Markov operators$\}$??
3. Every extreme point in $\mathscr{T}$ is non-atomic. ???

Thank you for your attention.

Questions and Comments?

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Further Research

- With respect to several measures of dependence, the class of non-atomic copulas is dense in the class of copulas.
- The only postulate on a measure of dependence that involves intermediate dependence (between independence and complete dependence) hinges on the Gaussian copulas.
- Our goal is to study how to impose a condition involving *factorizable* copulas on a general measure of dependence. (With T. Printechapat)