Tameness in the real field

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A **structure** $\mathcal{S}$ **on the real field** is a sequence of boolean algebras S_n of subsets of $\mathbb{R}^n$, for each $n = 1, 2, \ldots$, such that

1. $\Delta_{ij} := \{x \in \mathbb{R}^n : x_i = x_j\} \in S_n$ for $1 \leq i < j \leq n$
2. $A \in S_n \Rightarrow A \times \mathbb{R}, \mathbb{R} \times A \in S_{n+1}$
3. $A \in S_{n+1} \Rightarrow \pi(A) \in S_n$ where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection map $(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n)$
4. $\{(x, y) \in \mathbb{R}^2 : x < y\} \in S_2$
5. S_n contains all real algebraic subsets of $\mathbb{R}^n$.

Let $X \subseteq \mathbb{R}^n$. We say that X is **definable in** $\mathcal{S}$ if $X \in S_n$.

A **semialgebraic set** is a subset of $\mathbb{R}^n$ that is a finite boolean combination of sets of the form $\{x \in \mathbb{R}^n : f(x) = 0\}$ and $\{x \in \mathbb{R}^n : g(x) > 0\}$ where f and g are polynomials over the reals in $x_1, \ldots, x_n$.

The collection of semialgebraic sets forms the smallest structure on the real field.

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Examples

1. $(\mathbb{R}; +, \cdot, \mathbb{Z})$ NOT tame. "wild"
2. $(\mathbb{R}; +, \cdot)$ and $(\mathbb{R}; +, \cdot, \exp)$ tame $\Rightarrow$ o-minimal.

A structure is **o-minimal** if every unary definable set is a finite union of points and open intervals.

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Some properties of o-minimal structures

Suppose $\mathcal{S}$ is an o-minimal structure.

Cell Decomposition Theorem (van den Dries, Pillay-Steinhorn,1990s)

1. If $f\colon X \to \mathbb{R}$ ($X \subseteq \mathbb{R}^n$) is a definable function, then there is a finite partition of X so that $f\restriction_C$ is continuous for every C in the partition.
2. Every definable sets has finitely many connected components.

Definable Choice (van den Dries,1990s)

Let $\{S_x\}_{x\in X}$ be a definable family of subsets of $\mathbb{R}^n$. Then there exists a definable function $f\colon X \to \mathbb{R}^n$ such that $f(x) \in S_x$ for all $x \in X$.

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Recall

1. $(\mathbb{R}; +, \cdot, E)$ defines $\mathbb{Z} \Rightarrow$ WILD.
2. $(\mathbb{R}; +, \cdot, E)$ is o-minimal $\Rightarrow$ TAME.

Question

How about the intermediate steps?

Recall

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How about the intermediate steps?

Let $2^{\mathbb{Z}} = \{2^n : n \in \mathbb{Z}\}$.

- $(\mathbb{R}; +, \cdot, 2^{\mathbb{Z}})$ is not o-minimal but does not define $\mathbb{Z}$.
  Every unary definable set is a union of open intervals and finitely many discrete sets. $\Rightarrow$ **d-minimal**.
- There is a Cantor-like set C such that $(\mathbb{R}; +, \cdot, C)$ is not d-minimal and does not define $\mathbb{Z}$.
  Every unary definable set either has nonempty interior or is nowhere dense.

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Dimension Theory and Tameness

Let $\Pi(n, d)$ be the collection of the coordinates projections from $\mathbb{R}^n$ to $\mathbb{R}^d$. Let $E \subseteq \mathbb{R}^n$.

$\dim E$ is the least d such that there is $\pi \in \Pi(n, d)$ such that πE has nonempty interior.

Dimension Theory and Tameness

For $r > 0$, let $N_r(E) =$ the least number of open balls of radius at most r that covers E. The **assouad dimension of** E is the infimum of the set of $\alpha \geq 0$ such that

$$\{(r/R)^\alpha N_r(B_r(x) \cap E) : x \in E, 0 < r < R < +\infty\}$$

is bounded.

Dimension Theory and Tameness

(Hieronymi and Miller,2016)

Let $E \subseteq \mathbb{R}^n$. Then $(\mathbb{R}; +, \cdot, E)$ does not define $\mathbb{Z}$ if and only if all "metric dimension" coincides on closed definable sets.

Measure Theory and Tameness

(Tychonievich,2013)

Let $E \subseteq \mathbb{R}^n$ be a d-dimensional C^1 submanifold that is definable in $(\mathbb{R}; +, \cdot, 2^{\mathbb{Z}})$. Then

E has finite d-dimensional Hausdorff measure if and only if $\dim \operatorname{fr} E < \dim E$.

Measure Theoretic Tameness = "does not define compact d-dimension C^1 submanifold with infinite d-dimensional Hausdorff measure."

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More constructive results.

C^p zero set problem

Let $f : \mathbb{R}^n \to \mathbb{R}$.

$$Z(f) := \{x \in \mathbb{R}^n : f(x) = 0\}$$

If f is continuous, then $Z(f)$ is closed.

If $E \subseteq \mathbb{R}^n$ is closed, then the distance function to E is a continuous function whose zero set is E.

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Throughout, assume $E \subseteq \mathbb{R}^n$ is closed and $p \in \mathbb{N}$.

Question

Is there a C^p function f such that $Z(f) = E$?

Yes.

Attributed to H. Whitney

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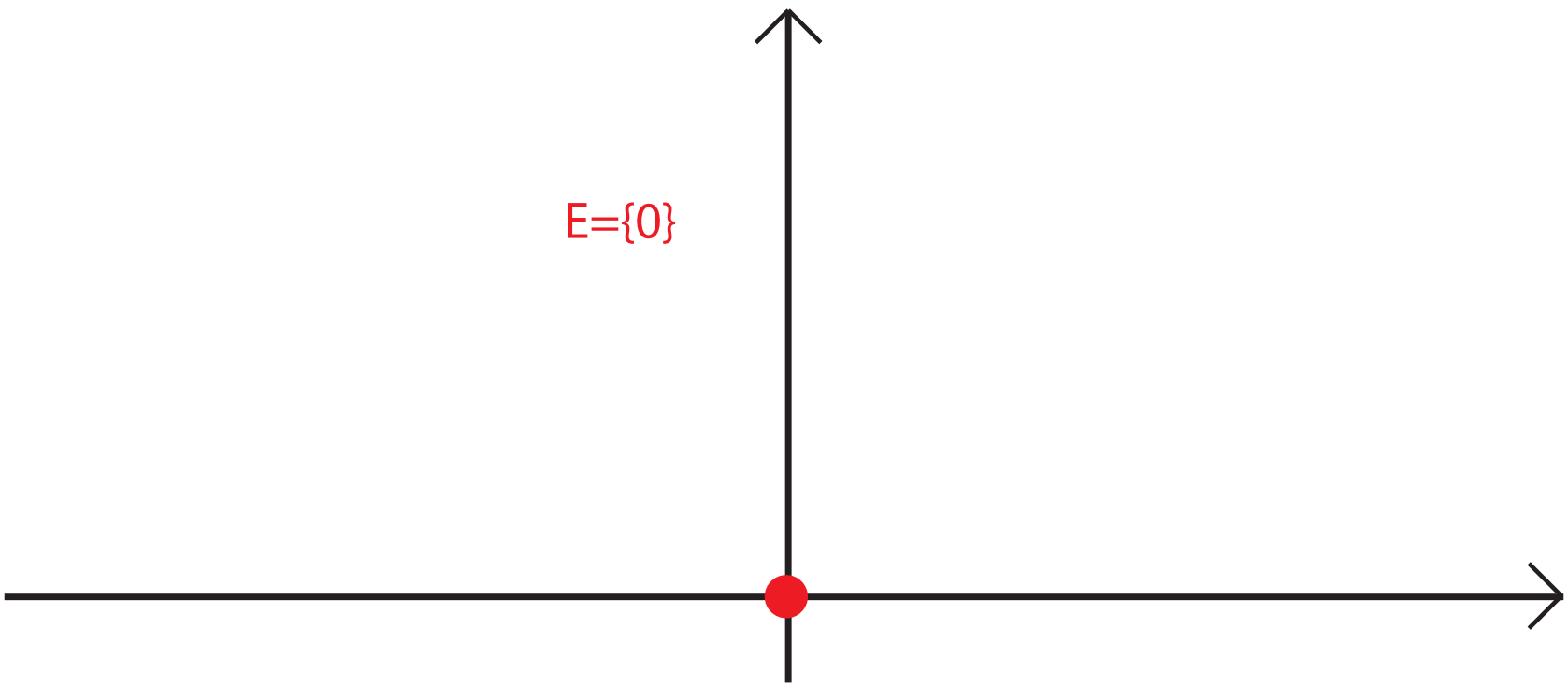

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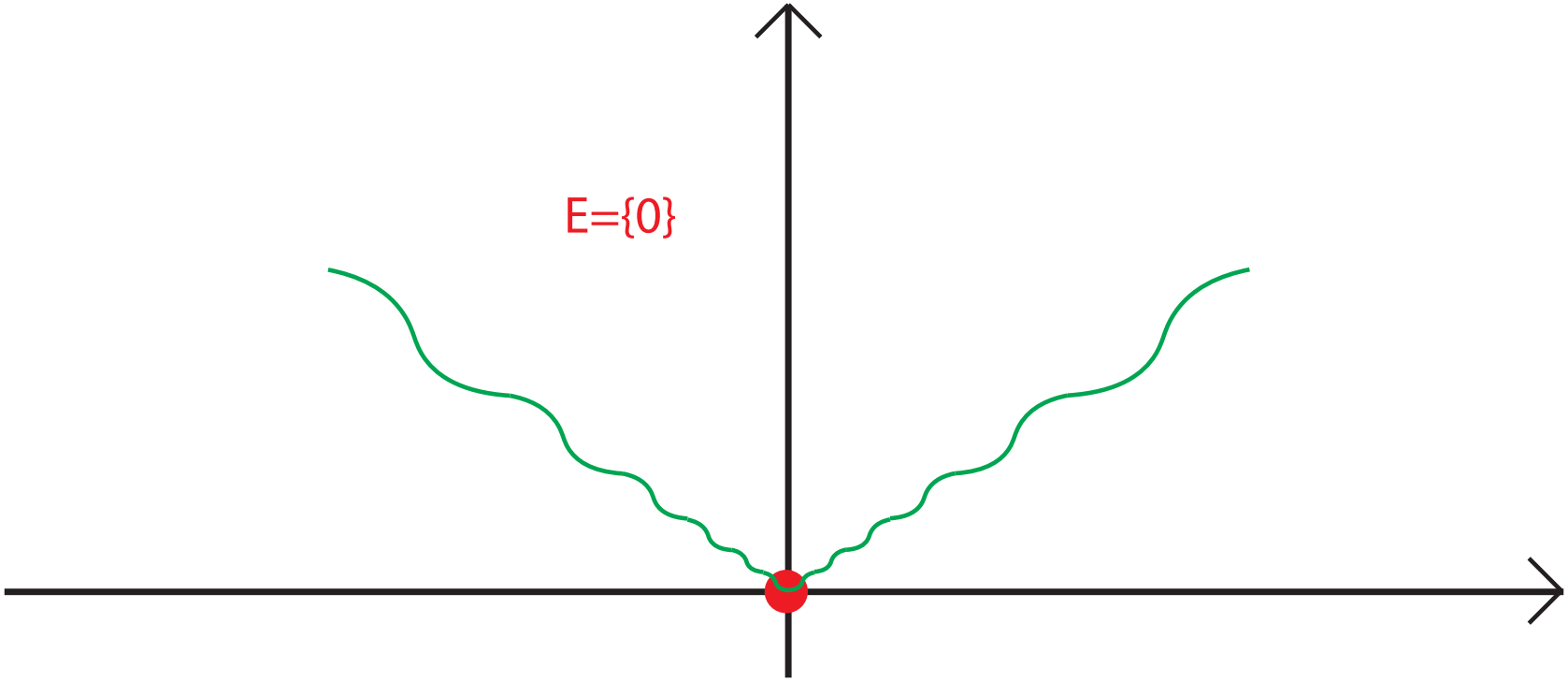

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C^p zero set problem

1. If $(\mathbb{R}, +, \cdot, E)$ defines $\mathbb{N}$, then YES. (Whitney)
2. If $(\mathbb{R}, +, \cdot, E)$ is o-minimal, then YES. (van den Dries & Miller)
3. If $(\mathbb{R}, +, \cdot, E)$ defines no infinite discrete sets, then question reduces to o-minimal case. (Miller & Speisseger,Tychonievich,Hieronymi)
4. If $(\mathbb{R}; +, \cdot, E)$ is d-minimal, then YES. (Miller & T.)

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Selection problem

Let $E \subseteq \mathbb{R}^n$ be closed and $T\colon E \rightrightarrows \mathbb{R}^m$ be a set-valued map from X to $\mathbb{R}^m$. Let $f\colon E \to \mathbb{R}^m$.
We say that f is a **selection of** T if $f(x) \in T(x)$ for every $x \in E$.

(Micheal's selection theorem)

If T is lower semi-continuous, then there is a continuous function f which is a selection of T.

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