Tameness in the real field

Athipat Thamrongthanyalak

Department of Mathematics and Computer Science
Faculty of Science, Chulalongkorn University

September 20, 2017
A structure $S$ on the real field is a sequence of boolean algebras $S_n$ of subsets of $\mathbb{R}^n$, for each $n = 1, 2, \ldots$, such that

1. $\Delta_{ij} := \{x \in \mathbb{R}^n : x_i = x_j\} \in S_n$ for $1 \leq i < j \leq n$
2. $A \in S_n \Rightarrow A \times \mathbb{R}, \mathbb{R} \times A \in S_{n+1}$
3. $A \in S_{n+1} \Rightarrow \pi(A) \in S_n$ where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection map $(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n)$
4. $\{(x, y) \in \mathbb{R}^2 : x < y\} \in S_2$
5. $S_n$ contains all real algebraic subsets of $\mathbb{R}^n$.

Let $X \subseteq \mathbb{R}^n$. We say that $X$ is **definable in** $S$ if $X \in S_n$. 
A **semialgebraic set** is a subset of $\mathbb{R}^n$ that is a finite boolean combination of sets of the form \( \{ x \in \mathbb{R}^n : f(x) = 0 \} \) and \( \{ x \in \mathbb{R}^n : g(x) > 0 \} \) where $f$ and $g$ are polynomials over the reals in $x_1, \ldots, x_n$.

The collection of semialgebraic sets forms the smallest structure on the real field.

For $E \subseteq \mathbb{R}^n$, let $(\mathbb{R}; +, \cdot, E)$ denote the smallest structure that contains $E$. 

Athipat Thamrongthanyalak  
Tameness in the real field
A semialgebraic set is a subset of $\mathbb{R}^n$ that is a finite boolean combination of sets of the form $\{x \in \mathbb{R}^n : f(x) = 0\}$ and $\{x \in \mathbb{R}^n : g(x) > 0\}$ where $f$ and $g$ are polynomials over the reals in $x_1, \ldots, x_n$.

The collection of semialgebraic sets forms the smallest structure on the real field.

For $E \subseteq \mathbb{R}^n$, let $(\mathbb{R}; +, \cdot, E)$ denote the smallest structure that contains $E$. 
Examples

1. \((\mathbb{R}; +, \cdot, \mathbb{Z})\) NOT tame. “wild”

2. \((\mathbb{R}; +, \cdot)\) and \((\mathbb{R}; +, \cdot, \exp)\) tame \(\Rightarrow\) o-minimal.

A structure is **o-minimal** if every unary definable set is a finite union of points and open intervals.

\(\Rightarrow\) Tame topology (Grothendieck)
Examples

1. \((\mathbb{R}; +, \cdot, \mathbb{Z})\) NOT tame. “wild”
2. \((\mathbb{R}; +, \cdot)\) and \((\mathbb{R}; +, \cdot, \exp)\) tame \(\Rightarrow\) o-minimal.

A structure is o-minimal if every unary definable set is a finite union of points and open intervals.

\(\Rightarrow\) Tame topology (Grothendieck)
Examples

1. \((\mathbb{R}; +, \cdot, \mathbb{Z})\) NOT tame. “wild”
2. \((\mathbb{R}; +, \cdot)\) and \((\mathbb{R}; +, \cdot, \exp)\) tame \(\Rightarrow\) o-minimal.

A structure is **o-minimal** if every unary definable set is a finite union of points and open intervals.

\(\Rightarrow\) Tame topology (Grothendieck)
Suppose $S$ is an o-minimal structure.

Cell Decomposition Theorem (van den Dries, Pillay-Steinhorn, 1990s)

1. If $f : X \to \mathbb{R}$ ($X \subseteq \mathbb{R}^n$) is a definable function, then there is a finite partition of $X$ so that $f \upharpoonright C$ is continuous for every $C$ in the partition.

2. Every definable sets has finitely many connected components.

Definable Choice (van den Dries, 1990s)

Let $\{S_x\}_{x \in X}$ be a definable family of subsets of $\mathbb{R}^n$. Then there exists a definable function $f : X \to \mathbb{R}^n$ such that $f(x) \in S_x$ for all $x \in X$. 
Suppose $S$ is an o-minimal structure.

**Cell Decomposition Theorem (van den Dries, Pillay-Steinhorn, 1990s)**

1. If $f : X \to \mathbb{R} (X \subseteq \mathbb{R}^n)$ is a definable function, then there is a finite partition of $X$ so that $f \upharpoonright C$ is continuous for every $C$ in the partition.
2. Every definable sets has finitely many connected components.

**Definable Choice (van den Dries, 1990s)**

Let $\{S_x\}_{x \in X}$ be a definable family of subsets of $\mathbb{R}^n$. Then there exists a definable function $f : X \to \mathbb{R}^n$ such that $f(x) \in S_x$ for all $x \in X$. 
Recall

1. $(\mathbb{R}; +, \cdot, E)$ defines $\mathbb{Z} \Rightarrow$ WILD.
2. $(\mathbb{R}; +, \cdot, E)$ is o-minimal $\Rightarrow$ TAME.

Question
How about the intermediate steps?
Recall

1. \((\mathbb{R}; +, \cdot, E)\) defines \(\mathbb{Z} \Rightarrow \text{WILD.}\)
2. \((\mathbb{R}; +, \cdot, E)\) is o-minimal \(\Rightarrow \text{TAME.}\)

Question

How about the intermediate steps?
Let $2^\mathbb{Z} = \{2^n : n \in \mathbb{Z}\}$.

- $(\mathbb{R}; +, \cdot, 2^\mathbb{Z})$ is not o-minimal but does not define $\mathbb{Z}$.
  Every unary definable set is a union of open intervals and finitely many discrete sets. \(\Rightarrow\) **d-minimal**.

- There is a Cantor-like set $C$ such that $(\mathbb{R}; +, \cdot, C)$ is not d-minimal and does not define $\mathbb{Z}$.
  Every unary definable set either has nonempty interior or is nowhere dense.
Let $2^\mathbb{Z} = \{2^n : n \in \mathbb{Z}\}$.

- $(\mathbb{R}; +, \cdot, 2^\mathbb{Z})$ is not o-minimal but does not define $\mathbb{Z}$. Every unary definable set is a union of open intervals and finitely many discrete sets. ⇒ d-minimal.

- There is a Cantor-like set $C$ such that $(\mathbb{R}; +, \cdot, C)$ is not d-minimal and does not define $\mathbb{Z}$. Every unary definable set either has nonempty interior or is nowhere dense.
Let $2^\mathbb{Z} = \{2^n : n \in \mathbb{Z}\}$.

- $(\mathbb{R}; +, \cdot, 2^\mathbb{Z})$ is not o-minimal but does not define $\mathbb{Z}$. Every unary definable set is a union of open intervals and finitely many discrete sets. $\Rightarrow$ d-minimal.

- There is a Cantor-like set $C$ such that $(\mathbb{R}; +, \cdot, C)$ is not d-minimal and does not define $\mathbb{Z}$. Every unary definable set either has nonempty interior or is nowhere dense.
Let $2^\mathbb{Z} = \{2^n : n \in \mathbb{Z}\}$.

- $(\mathbb{R}; +, \cdot, 2^\mathbb{Z})$ is not o-minimal but does not define $\mathbb{Z}$. Every unary definable set is a union of open intervals and finitely many discrete sets. $\Rightarrow$ **d-minimal**.

- There is a Cantor-like set $C$ such that $(\mathbb{R}; +, \cdot, C)$ is not d-minimal and does not define $\mathbb{Z}$.

  Every unary definable set either has nonempty interior or is nowhere dense.
Let $2\mathbb{Z} = \{2^n : n \in \mathbb{Z}\}$.

- $(\mathbb{R}; +, \cdot, 2\mathbb{Z})$ is not o-minimal but does not define $\mathbb{Z}$. Every unary definable set is a union of open intervals and finitely many discrete sets. $\Rightarrow$ **d-minimal**.

- There is a Cantor-like set $C$ such that $(\mathbb{R}; +, \cdot, C)$ is not d-minimal and does not define $\mathbb{Z}$. Every unary definable set either has nonempty interior or is nowhere dense.
Scope of the research program

- The tameness hierarchy.
- Classification of sets in the hierarchy.
- Properties of each class in the hierarchy.
The tameness hierarchy.
Classification of sets in the hierarchy.
Properties of each class in the hierarchy.
Scope of the research program

- The tameness hierarchy.
- Classification of sets in the hierarchy.
- Properties of each class in the hierarchy.
Let $\Pi(n, d)$ be the collection of the coordinates projections from $\mathbb{R}^n$ to $\mathbb{R}^d$. Let $E \subseteq \mathbb{R}^n$.

$\dim E$ is the least $d$ such that there is $\pi \in \Pi(n, d)$ such that $\pi E$ has nonempty interior.
For $r > 0$, let $N_r(E)$ be the least number of open balls of radius at most $r$ that covers $E$. The **assouad dimension of** $E$ is the infimum of the set of $\alpha \geq 0$ such that

$$\{(r/R)^\alpha N_r(B_r(x) \cap E) : x \in E, 0 < r < R < +\infty\}$$

is bounded.
Let $E \subseteq \mathbb{R}^n$. Then $(\mathbb{R}; +, \cdot, E)$ does not define $\mathbb{Z}$ if and only if all “metric dimension” coincides on closed definable sets.

(Hieronymi and Miller, 2016)
Let $E \subseteq \mathbb{R}^n$ be a $d$-dimensional $C^1$ submanifold that is definable in $(\mathbb{R}; +, \cdot, 2\mathbb{Z})$. Then

$$E \text{ has finite } d\text{-dimensional Hausdorff measure if and only if } \dim \text{ fr } E < \dim E.$$
Let $E \subseteq \mathbb{R}^n$ be a $d$-dimensional $C^1$ submanifold that is definable in $(\mathbb{R}; +, \cdot, 2^{\mathbb{Z}})$. Then

$E$ has finite $d$-dimensional Hausdorff measure if and only if $\dim \text{fr } E < \dim E$.

Measure Theoretic Tameness = “does not define compact $d$-dimension $C^1$ submanifold with infinite $d$-dimensional Hausdorff measure.”

Measure Theoretic Tameness provides a different tameness hierarchy.
(Tychonievich, 2013)

Let $E \subseteq \mathbb{R}^n$ be a $d$-dimensional $C^1$ submanifold that is definable in $(\mathbb{R}; +, \cdot, 2^\mathbb{Z})$. Then

$$E \text{ has finite } d\text{-dimensional Hausdorff measure if and only if } \dim \text{fr } E < \dim E.$$ 

Measure Theoretic Tameness = “does not define compact $d$-dimension $C^1$ submanifold with infinite $d$-dimensional Hausdorff measure.”

(T, 2017)

Measure Theoretic Tameness provides a different tameness hierarchy.
More constructive results.
Let $f: \mathbb{R}^n \to \mathbb{R}$.

$$Z(f) := \{ x \in \mathbb{R}^n : f(x) = 0 \}$$

If $f$ is continuous, then $Z(f)$ is closed.

If $E \subseteq \mathbb{R}^n$ is closed, then the distance function to $E$ is a continuous function whose zero set is $E$. 
Let $f : \mathbb{R}^n \to \mathbb{R}$.

$$Z(f) := \{x \in \mathbb{R}^n : f(x) = 0\}$$

If $f$ is continuous, then $Z(f)$ is closed.

If $E \subseteq \mathbb{R}^n$ is closed, then the distance function to $E$ is a continuous function whose zero set is $E$. 
Let $f : \mathbb{R}^n \to \mathbb{R}$.

$$Z(f) := \{ x \in \mathbb{R}^n : f(x) = 0 \}$$

If $f$ is continuous, then $Z(f)$ is closed.

If $E \subseteq \mathbb{R}^n$ is closed, then the distance function to $E$ is a continuous function whose zero set is $E$. 
Throughout, assume $E \subseteq \mathbb{R}^n$ is closed and $p \in \mathbb{N}$.

**Question**

Is there a $C^p$ function $f$ such that $Z(f) = E$?

Yes.

**Attributed to H. Whitney**

There is a $C^p$ function $f : \mathbb{R}^n \to \mathbb{R}$ such that $Z(f) = E$. 
Throughout, assume $E \subseteq \mathbb{R}^n$ is closed and $p \in \mathbb{N}$.

**Question**

Is there a $C^p$ function $f$ such that $Z(f) = E$?

Yes.

**Attributed to H. Whitney**

There is a $C^p$ function $f : \mathbb{R}^n \to \mathbb{R}$ such that $Z(f) = E$. 
$C^p$ zero set problem

$E = \{0\}$

Athipat Thamrongthanyalak

Tameness in the real field
$C^p$ zero set problem

$E = \{0\}$
$C^p$ zero set problem

**Questions**

- If $E$ is well behaved in some prescribed sense, can $f$ be chosen to be equally well behaved?
- Is there a $C^p$ function definable in $(\mathbb{R}, +, \cdot, E)$ whose zero set is $E$?
Questions

- If $E$ is well behaved in some prescribed sense, can $f$ be chosen to be equally well behaved?
- Is there a $C^p$ function definable in $(\mathbb{R}, +, \cdot, E)$ whose zero set is $E$?
1. If $(\mathbb{R}, +, \cdot, E)$ defines $\mathbb{N}$, then YES.
(Whitney)

2. If $(\mathbb{R}, +, \cdot, E)$ is o-minimal, then YES.
(van den Dries & Miller)

3. If $(\mathbb{R}, +, \cdot, E)$ defines no infinite discrete sets, then question reduces to o-minimal case.
(Miller & Speissegger, Tychonievich, Hieronymi)

4. If $(\mathbb{R}; +, \cdot, E)$ is d-minimal, then YES.
(Miller & T.)
If \((\mathbb{R}, +, \cdot, E)\) defines \(\mathbb{N}\), then YES. (Whitney)

If \((\mathbb{R}, +, \cdot, E)\) is o-minimal, then YES. (van den Dries & Miller)

If \((\mathbb{R}, +, \cdot, E)\) defines no infinite discrete sets, then question reduces to o-minimal case. (Miller & Speissegger, Tychonievich, Hieronymi)

If \((\mathbb{R}; +, \cdot, E)\) is d-minimal, then YES. (Miller & T.)
1. If $(\mathbb{R}, +, \cdot, E)$ defines $\mathbb{N}$, then YES. (Whitney)

2. If $(\mathbb{R}, +, \cdot, E)$ is o-minimal, then YES. (van den Dries & Miller)

3. If $(\mathbb{R}, +, \cdot, E)$ defines no infinite discrete sets, then question reduces to o-minimal case. (Miller & Speissegger, Tychonievich, Hieronymi)

4. If $(\mathbb{R}; +, \cdot, E)$ is d-minimal, then YES. (Miller & T.)
1. If $(\mathbb{R}, +, \cdot, E)$ defines $\mathbb{N}$, then YES. (Whitney)
2. If $(\mathbb{R}, +, \cdot, E)$ is o-minimal, then YES. (van den Dries & Miller)
3. If $(\mathbb{R}, +, \cdot, E)$ defines no infinite discrete sets, then question reduces to o-minimal case. (Miller & Speisseegeger, Tychonieichvich, Hieronymi)
4. If $(\mathbb{R}; +, \cdot, E)$ is d-minimal, then YES. (Miller & T.)
Let $E \subseteq \mathbb{R}^n$ be closed and $T : E \Rightarrow \mathbb{R}^m$ be a set-valued map from $X$ to $\mathbb{R}^m$. Let $f : E \to \mathbb{R}^m$.

We say that $f$ is a **selection of** $T$ if $f(x) \in T(x)$ for every $x \in E$.

(Michael’s selection theorem)

If $T$ is lower semi-continuous, then there is a continuous function $f$ which is a selection of $T$. 

Athipat Thamrongthanyalak

Tameness in the real field
Let $E \subseteq \mathbb{R}^n$ be closed and $T: E \rightrightarrows \mathbb{R}^m$ be a set-valued map from $X$ to $\mathbb{R}^m$. Let $f: E \rightarrow \mathbb{R}^m$.

We say that $f$ is a selection of $T$ if $f(x) \in T(x)$ for every $x \in E$.

(Micheal’s selection theorem)

If $T$ is lower semi-continuous, then there is a continuous function $f$ which is a selection of $T$. 
Selection problem

Questions

- If $T$ is well behaved in some prescribed sense, can $f$ be chosen to be equally well behaved?
- Is there a continuous selection of $T$ which is definable in $(\mathbb{R}, +, \cdot, T)$?
Questions

- If $T$ is well behaved in some prescribed sense, can $f$ be chosen to be equally well behaved?
- Is there a continuous selection of $T$ which is definable in $(\mathbb{R}, +, \cdot, T)$?
Selection problem

1. If $\langle \mathbb{R}, +, \cdot, T \rangle$ defines $\mathbb{N}$, then YES. (Whitney)
2. If $\langle \mathbb{R}, +, \cdot, T \rangle$ is o-minimal, then YES. (van den Dries & Miller)
3. If $\langle \mathbb{R}, +, \cdot, T \rangle$ defines no infinite discrete sets, then question reduces to o-minimal case. (Miller & Speissegger, Tychonievich, Hieronymi)
4. If $\langle \mathbb{R}; +, \cdot, T \rangle$ is d-minimal, then YES. (Miller & T.)
1. If \((\mathbb{R}, +, \cdot, T)\) defines \(\mathbb{N}\), then YES.
(Whitney)

2. If \((\mathbb{R}, +, \cdot, T)\) is o-minimal, then YES.
(van den Dries & Miller)

3. If \((\mathbb{R}, +, \cdot, T)\) defines no infinite discrete sets, then question reduces to o-minimal case.
(Miller & Speissegger, Tychonievich, Hieronymi)

4. If \((\mathbb{R}; +, \cdot, T)\) is d-minimal, then YES.
(Miller & T.)
1 If $(\mathbb{R}, +, \cdot, T)$ defines $\mathbb{N}$, then YES. (Whitney)

2 If $(\mathbb{R}, +, \cdot, T)$ is o-minimal, then YES. (van den Dries & Miller)

3 If $(\mathbb{R}, +, \cdot, T)$ defines no infinite discrete sets, then question reduces to o-minimal case. (Miller & Speissegger, Tychonievich, Hieronymi)

4 If $(\mathbb{R}; +, \cdot, T)$ is d-minimal, then YES. (Miller & T.)
1. If \((\mathbb{R}, +, \cdot, T)\) defines \(\mathbb{N}\), then YES. (Whitney)

2. If \((\mathbb{R}, +, \cdot, T)\) is o-minimal, then YES. (van den Dries & Miller)

3. If \((\mathbb{R}, +, \cdot, T)\) defines no infinite discrete sets, then question reduces to o-minimal case. (Miller & Speissegger, Tychonievich, Hieronymi)

4. If \((\mathbb{R}; +, \cdot, T)\) is d-minimal, then YES. (Miller & T.)
Thank you.