Assignment 7

1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 1$ for all $x \in \mathbb{R}$. Find $f[(-1, 2)]$ and $f^{-1}[(0, 2]]$.

2. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \lfloor x \rfloor$. Determine $f[(0, 2]], f^{-1}[\{0\}]$ and $f^{-1}[[-2, 1)]$.

3. Let $f : X \to Y, g : Y \to Z, A \subseteq X$ and $B \subseteq Y$. Show that $g[B] \cap (X \setminus g \circ f[A]) \subseteq g[B \setminus f[A]]$.

4. Prove that $\mathbb{R} \approx (0, 1)$.

5. Prove that any two nonempty open interval in $\mathbb{R}$ are similar.

6. Prove that $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$.

7. Determine whether each of the following sets is finite, denumerable or uncountable.

   7.1 $\{x \in \mathbb{Q} | 1 < x < 2\}$.

   7.2 $\{\frac{m}{n} | m, n \in \mathbb{N}, m < 50 \text{ and } 5 < n < 100\}$.

   7.3 $\{(a, b) \in \mathbb{Q} \times \mathbb{Q} | a + b = 1\}$.

   7.4 $\{(a, b) \in \mathbb{R} \times \mathbb{R} | b = \sqrt{1 - a^2}\}$.

Assignment 6

1. Determine whether or not each of the following defines a one-to-one and/or an onto function. If any function is not onto, find (with proof) its range.

   1.1 $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}; f(x, y) = 2x + 3y$.

   1.2 $f : \mathbb{R} \to \mathbb{R}; f(x) = x + \lfloor x \rfloor$.

   1.3 $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}; f(x, y) = x^2 + y^2 + 1$.

2. Prove that there is a bijection from $\mathbb{Z}$ to $\mathbb{N}$.

3. Prove that there is an onto function from $\mathbb{N}$ to $\mathbb{N}$ which is not one-to-one.

4. Define $f : \mathbb{Z}_n \to \mathbb{Z}$ by $f([a]) = 2a$ for all $[a] \in \mathbb{Z}_n$. Is f well-defined? Justify your answer.

5. Do the same as Exercise 4. for $f : \mathbb{Z}_n \to \mathbb{Z}_n$ defined by $f([a]) = [n - a]$

6. Define $f : \mathbb{Z} \to \mathbb{N}$ by $f(x) = \begin{cases} 2|x| & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0. \end{cases}$

   6.1 Show that f is a bijection.

   6.2 Determine a closed form of $f^{-1}(x)$ for any $x \in \mathbb{N}$.

7. Let A be a set and $a \in A$. For a function $f : A \to \{1, 2, \ldots, n\}$ where $n \in \mathbb{N}$, define $g_f : \{1, 2, \ldots, n\} \to A$ by

$$g_f(x) = \begin{cases} f^{-1}(x) & \text{if } x \in f[A], \\ a & \text{if } x \notin f[A]. \end{cases}$$

   7.1 Give an example to show that g is not necessarily well-defined.

   7.2 Give a condition on f so that g is well-defined, and show that in this case, g_f is an onto function.

8. Show that the decimal expansion of a rational number must, after some point, become periodic or stop.

9. In a room where there are more than 51 people with age between 1 and 100, show that either two people have the same age or there are 2 people whose ages are consecutive intergers.

Assignment 5

1. For each of the following relations, determine whether the relation is reflexive, symmetric, antisymmetric or transitive.
   1.1 $R \subseteq \mathbb{Z} \times \mathbb{Z}$, $xRy \Leftrightarrow x + y$ is even.
   1.2 R on $\mathbb{Z} \times \mathbb{Z}, (a, b)R(c, d) \Leftrightarrow a \leq c$.
   1.3 R on $\mathbb{R} \times \mathbb{R}, (a, b)R(c, d) \Leftrightarrow a - b = c - d$.
   1.4 R on $\mathbb{R}, xRy \Leftrightarrow x - y \in \mathbb{Z}$.

2. Which relations in Exercise 1. are partial orders? Which are equivalence relations. For each equivalence relation, give examples of equivalence classes.

3. Let R and S be relations on a nonempty set A. Prove or disprove each of the following conjectures :
   3.1 R and S are transitive $\Rightarrow R \cup S$ is transitive.
   3.2 R and S are reflexive $\Rightarrow R \circ S$ is reflexive.
   3.2 R and S are symmetric $\Rightarrow R \cup S$ is symmetric.

4. Let $(A, \preccurlyeq_A)$ and $(B, \preccurlyeq_B)$ be well-ordered sets. Define $\preccurlyeq$ on $A \cup B$ as follows : for any $x, y \in A \cup B$

   $$x \preccurlyeq y \text{ iff one of the following holds}$$
   $$\text{(i) } x \in A, y \in A \text{ and } x \preccurlyeq_A y$$
   $$\text{(ii) } x \in B, y \in B \text{ and } x \preccurlyeq_B y$$
   $$\text{(iii) } x \in A \text{ and } y \in B.$$

   Show that $(A \cup B, \preccurlyeq)$ is a well-ordered set.

5. Let $(P, \preccurlyeq)$ be a poset. prove or disprove the following statements.
   5.1 If $(P, \preccurlyeq)$ is a lattice, then it is total order.
   5.2 If $(P, \preccurlyeq)$ is a total order, then it is a lattice.

6. Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6\}$ and $A = \mathcal{P}(\mathcal{U})$. Then A is a poset under inclusion. Let $B = \{\{1\}, \{2\}, \{2, 3\}, \{1, 4\}\}$.
   6.1 Determine whether A is totally ordered or well-ordered.
   6.2 Is A a lattice? Is B a lattice?
   6.3 Determine the number of upper bounds for B. Find the least upper bound of B.
   6.4 Give an example of a maximal chain in A.

4. Let $\emptyset \neq A \subseteq B \subseteq \mathbb{R}$. Show that
   If B is bounded above then so is A and $\sup A \leq \sup B$.
   If B is bounded below then so is A and $\inf B \leq \inf A$.

5. Assume the supremum property of $\mathbb{R}$, prove the infimum property of $\mathbb{R}$.

Definition. Let S be a subset of $\mathbb{R}$

(i) $u \in \mathbb{R}$ is said to be an **upper bound** of S if $s \leq u$ for all $s \in S$.

(ii) $w \in \mathbb{R}$ is said to be a **lower bound** of S if $w \leq s$ for all $s \in S$.

(iii) S is **bounded above** if S has an upper bound.

(iv) S is **bounded below** if S has a lower bound.

(v) S is **bounded** if S is bounded above and bounded below.
S is **unbounded** if it lacks either an upper bound or a lower bound.

Definition. Let S be a subset of $\mathbb{R}$

(i) If S is bounded above, then u_0 is said to be a **supremum** (a least upper bound) of S if no number smaller than u_0 is an upper bound of S. u_0 is then denoted $\sup S$.

(ii) If S is bounded below, then w_0 is said to be an **infimum** (a greatest lower bound) of S if no number larger than w_0 is an lower bound of S. w_0 is then denoted $\inf S$.

The Supremum Property of $\mathbb{R}$. Every nonempty set of real numbers which is bounded above has a supremum in $\mathbb{R}$.

The Infimum Property of $\mathbb{R}$. Every nonempty set of real numbers which is bounded below has an infimum in $\mathbb{R}$.

Assignment 4

1. Let u be an upper bound of a nonempty subset S of $\mathbb{R}$. Show that $u = \sup S$ if and only if for each $\epsilon > 0$ there exists an $s_\epsilon \in S$ such that $u - \epsilon < s_\epsilon$.

2. Let $S = \{1 - \frac{(-1)^n}{n} \mid n \in \mathbb{N}\}$. Find $\inf S$ and $\sup S$.

3. Show that a nonempty finite subset of $\mathbb{R}$ contains its supremum and infimum.

Assignment 3

1. Show that between two distinct real numbers there is a nonzero rational number.

2. For each $k \in \mathbb{Z}$, let $\mathbb{Z}(k) = \{n \in \mathbb{Z} | \; n \geq k\}$. Prove that every nonempty subset of $\mathbb{Z}(k)$ has a smallest element.

3. For each A_n defined below, determine (with proof) $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$

   3.1 $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$
   3.2 $A_n = \left[1 - \frac{1}{n}, 2 + \frac{1}{n}\right)$
   3.3 $A_n = \left(1 - \frac{1}{n}, n\right)$
   3.4 $A_n = \left[1 + \frac{1}{n}, n + 1\right]$

4. Determine the following limits and justify your answer directly from the definitions.

   4.1 $\lim_{x \to 3} 2x - 1$
   4.2 $\lim_{x \to 2} x^2 + 1$
   4.3 $\lim_{x \to 1} x^3 + 2$
   4.4 $\lim_{n \to \infty} \frac{n}{n^2 + 1}$

Assignment 2

1. Show that a set with n elements has 2^n subsets.

2. Prove that each positive integer can be written in the form $2^k m$ for some nonnegative integer k and odd integer m.

3. Conjecture a formula for the sum

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)}$$

and check your conjecture by using mathematical induction.

4. Let S be a subset of $\mathbb{N}$ such that

   (a) $a^k \in S$ for all $k \in \mathbb{N}$, and

   (b) if $k \in S$ and $k \geq 2$, then $k - 1 \in S$.

   Prove that $S = \mathbb{N}$.

5. Show that for each $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$, $a^{m+n} = a^m \cdot a^n$ and $(a^m)^n = a^{mn}$.

Assignment 1

1. Translate each of the following statements into logical expressions and write its negation.

   1.1) The product of any two consecutive integers is even.
   1.2) There is a real number y such that $y < x$ for every real number x.
   1.3) For every real number x, there exits a real number y such that $y < x$.
   1.4) There exists a real number whose square is 5.
   1.5) The sum of a nonzero rational number and an irrational number is irrational.
   1.6) The sum of square of two real numbers is nonnegative.
   1.7) There exists a real number y such that $xy < 1$ for all real number x.
   1.8) Product of two irrational numbers is an irrational number.
   1.9) There is a smallest rational number.
   1.10) The set $\{x \in \mathbb{R} \mid x^2 + x + 1 = 0\}$ is empty.

2. Prove or disprove each of the above statements.

3. Prove that $n^2 + n - 2$ is even for all integer n.

4. Let $a, b \in \mathbb{Z}$. Prove that ab is even if and only if a is even or b is even.

5. Determine the true value of each statement.

   5.1 $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x = y + 1$.
   5.2 $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy < 2$.

6. Let a and b be real numbers. Show that

   6.1 If $a \geq 0$ and $b \geq 0$, then $a < b \Leftrightarrow a^2 < b^2$.
   6.2 If $ab > 0$, then either
   (i) $a > 0$ and $b > 0$, or
   (ii) $a < 0$ and $b < 0$.
   6.3 If $ab < 0$, then either
   (i) $a < 0$ and $b > 0$, or
   (ii) $a > 0$ and $b < 0$.