Assignment 1

1. Let $R_1, R_2, \ldots, R_n$ be rings. Prove that

   (i) $R_1 \times R_2 \times \ldots \times R_n$ is a ring under componentwise addition and multiplication.

   (ii) $R_1 \times R_2 \times \ldots \times R_n$ is commutative if and only if R_i is commutative for all $i = 1, 2, \ldots, n$.

   (iii) $(e_1, e_2, \ldots, e_n)$ is an identity of $R_1 \times R_2 \times \ldots \times R_n$ if and only if e_i is an identity of R_i for all i.

2. Give an example of an infinite noncommutative ring without identity.

   Give an example of an finite noncommutative ring without identity.

3. Prove that a ring can have at most one identity.

4. In each case show that S is a subring of $M_2(\mathbb{R})$

   4.1 $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}) \middle| \, a + c = b + d \right\}.$

   4.2 $S = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$

   4.3 $S = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$

   4.4 $S = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| \, a, b, d \in \mathbb{R} \right\}.$

5. If S and T are subrings of R, show that $S \cap T$ is a subring of R.

6. Let X be a nonempty subset of a ring R and

$$C(X) = \{a \in R \,|\, ax = xa \text{ for all } x \in X\}.$$

   Show that $C(X)$ is a subring of R. It is called the **centerizer** of X in R. The **center** of R, denoted $Z(R)$, is the centerizer of R in R.

Assignment 2

1. Show that $\mathcal{F}(\mathbb{R})$ is not an integral domain and determine $\mathcal{U}(\mathcal{F}(\mathbb{R}))$.

2. Show that $\mathcal{U}(\mathbb{Z}_n) = \{\bar{a} \in \mathbb{Z}_n \,|\, g.c.d(a, n) = 1\}$.

3. Show that $\mathcal{U}(\mathbb{Z}[i]) = \{1, -1, i, -i\}$.

   Definition. An element e in a ring R is called an **idempotent** if $e^2 = e$.

4. Let R be a ring with 1 and e is an idempotent in R. Show that

   4.1 $(1 - e)$ is also an idempotent in R.

   4.2 If $e \neq 0$ or 1, then e is a zero divisor of R.

5. Let e be an idempotent of a ring R. Show that

$$eRe = \{ere \,|\, r \in R\}$$

   is a ring with identity e and $eRe = \{a \in R \,|\, ea = a = ae\}$.

   Definition. An element a in a ring R is called a **nilpotent** element if $a^n = 0$ for some positive integer n.

6. Show that the set of nilpotent elements of a commutative ring form a subring of R.

7. Show that 0 is the only nilpotent element in an integral domain.

8. Let $R = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}\}$ be a subset of $\mathbb{Z}_{10}$. Prove that R is a field under addition and multiplication modulo 10.

9. Find the units, the zero divisors, the nilpotents and the idempotents of each of the following rings :

   9.1 $\mathbb{Z}_9$ 9.2 $\mathbb{Z}_{12}$ 9.3 $\begin{bmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{bmatrix}$.

Assignment 3

1. Show that a finite ring must have a nonzero characteristic.
2. Let x and y be elements of an integral domain of prime characteristic p. Show that

   2.1 $(x+y)^p = x^p + y^p$

   2.2 $(x+y)^{p^k} = x^{p^k} + y^{p^k}$ for all $k \in \mathbb{N}$.
3. Let R be a ring with 1 and char R is finite. Show that char $R = \circ(u)$, the order of u under addition, for all unit u in R.

Assignment 4

1. Prove that the intersection of any set of ideals is an ideal.

2. Give an example to show that the union of two ideals need not be an ideal.

3. Let R be a ring with 1 and $a \in R$. Show that

   3.1 $Ra = \{ra \mid r \in R\}$ is the smallest left ideal of R containing a and $aR = \{ar \mid r \in R\}$ is the smallest right ideal of R containing a.

   3.2 If I is an ideal of R and $1 \in I$, then $I = R$.

4. Let I and J be ideals of a ring R and

$$I + J = \{a + b \mid a \in I \text{ and } b \in J\}$$

$$IJ = \{a_1b_1 + a_2b_2 + \ldots + a_nb_n \mid a_i \in A, b_i \in B, n \in \mathbb{N}\}$$

   Show that

   (i) $I + J$ and IJ are ideals of R

   (ii) $IJ \subseteq I \cap J$.

5. In the ring of integers $\mathbb{Z}$, find a positive integer a such that

   5.1 $a\mathbb{Z} = 3\mathbb{Z} + 4\mathbb{Z}$ 5.2 $a\mathbb{Z} = 4\mathbb{Z} + 6\mathbb{Z}$

   5.3 $a\mathbb{Z} = 3\mathbb{Z} \cdot 4\mathbb{Z}$ 5.4 $a\mathbb{Z} = 4\mathbb{Z} \cdot 6\mathbb{Z}$

Assignment 5

1. In each case determine whether I is an ideal of the ring R.

   1.1 $R = \mathbb{Z} \times \mathbb{Z}$, $I = \{(k,k) \mid k \in \mathbb{Z}\}$.

   1.2 $R = \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{bmatrix}$, $I = \begin{bmatrix} \mathbb{Z} & \mathbb{R} \\ 0 & \mathbb{Z} \end{bmatrix}$.

   1.3 $R = \mathbb{Z}[i]$, $I = \{n + ni \mid n \in \mathbb{Z}\}$.

   1.4 $R = \begin{bmatrix} S & S \\ 0 & S \end{bmatrix}$ where S is a ring, $I = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}$.

   1.5 $R = M_2(\mathbb{Z}_2)$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R$.

2. Describe the quotient ring $R/_I$ for each ideal I in the above problem.

3. Let n be an integer, show that

$$nR = \{nr \mid r \in R\} \text{ and } A_n \qquad = \{r \in R \mid nr = 0\}$$

   are ideals of R for any ring R.

4. Let I be an ideal of a ring R. Show that $I \cap S$ is an ideal of S for all subring S of R.

5. Let I be an ideal of R. Show that $R/_I$ is commutative if and only if $rs - sr \in I$ for all $r, s \in R$. Give an example where $R/_I$ is commutative but R is not.

6. Let R be a ring of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $I = \{f \in R \mid f(0) = 0\}$ is an ideal of R.

7. Find all maximal ideal of $\mathbb{Z}/_{10\mathbb{Z}}$.

8. Let R and S be rings with 1. Show that every ideal of $R \times S$ has that form $A \times B$ where A and B are ideals of R and S, respectively.

Assignment 6

1. Let R and S be rings. Show that the rings $R \times S$ and $S \times R$ are isomorphic.

2. Let R be a commutative ring of prime characteristic p. Show that the map $f : R \to R$ defined by

$$f(x) = x^p \qquad \text{for all } x \in R$$

is a ring homomorphism from R to R.

3. Show that if m and n are distinct positive integers, then $m\mathbb{Z}$ is not ring isomorphic to $n\mathbb{Z}$.

4. Show that the field of quotient of $\mathbb{Z}[\,i\,]$ is isomorphic to $\mathbb{Q}[\,i\,] = \{r + si \,|\, r, s \in \mathbb{Q}\}$.

5. In each case determine whether the map f is a ring homomorphism.

   5.1 $f : \mathbb{Z}_3 \to \mathbb{Z}_{12}$, where $f(\overline{a}) = \overline{4a}$

   5.2 $f : \mathbb{Z}_4 \to \mathbb{Z}_{12}$, where $f(\overline{a}) = \overline{3a}$

   5.3 $f : R \times R \to R$ where R is a ring and $f(r, s) = r + s$.

Assignment 7

1. Factor $f(x)$ in each case as a product of irreducible polynomial in $\mathbb{F}[x]$.

   1.1 $f(x) = x^6 - 1 \quad ; \ \mathbb{F} = \mathbb{Z}_3$.

   1.2 $f(x) = x^4 + 12 \quad ; \ \mathbb{F} = \mathbb{Z}_{13}$.

   1.3 $f(x) = x^4 - x^3 - x^2 - x - 2 \quad ; \ \mathbb{F} = \mathbb{Q}$.

2. Write $g(x) = f(x)q(x) + r(x)$ in $R[x]$, where $r(x) = 0$ or deg $r(x) <$ deg $f(x)$.

   2.1 $g(x) = x^3 + x^2 + 3x + 2 \quad ; \ f(x) = 3x + 1 \quad ; \ R = \mathbb{Z}_8$.

   2.2 $g(x) = 3x^3 + 5x^2 + x + 6 \quad ; \ f(x) = 2x^2 + 1 \quad ; \ R = \mathbb{Q}$.

3. Let A be an ideal of a ring R with 1 and

$$\bar{A} = \{a_0 + r_1x + r_2x^2 + ... + r_nx^n \,|\, n \in \mathbb{N}, a_0 \in A, r_i \in R\}.$$

   Show that $\bar{A}$ is an ideal of $R[x]$ and $R[x]/_{\bar{A}} \cong R/_A$

4. Determine the multiplicity of a as a root of $f(x)$.

   4.1 $f(x) = x^4 + 2x^2 + 2x + 2 \ ; \ a = -1 \ ; \ R = \mathbb{Z}_3$.

   4.2 $f(x) = x^3 - 2x^2 - 4x + 3 \ ; a = 3 \ ; \ R = \mathbb{Z}_6$.

5. Show that $f(x) = x^3 - 2x^2 + 3x - 2$ is reducible over any.

6. In each case find $\gcd(f(x), g(x))$ and express it in $F[x]$ as a linear combination of $f(x)$ and $g(x)$.

   6.1 $f(x) = x^2 + 2, g(x) = x^3 + 4x^2 + x + 1 \ ; \ F = \mathbb{Z}_5$.

   6.2 $f(x) = x^2 - x - 2, g(x) = x^5 - 4x^3 - 2x^2 - 7x - 6 \ ; \ F = \mathbb{Q}$.