Assigment 1

1. Show that the set $G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \middle| \; a, b, c \in \mathbb{R} \right\}$ is a group under matrix multiplication.

2. Let $\mathcal{U}$ be a set and $G = \{A \mid A \subseteq \mathcal{U}\}$. Show that G ia an abelian group under the operation $\oplus$ defined by

$$A \oplus B = (A \smallsetminus B) \cup (B \smallsetminus A).$$

3. In each case, determine whether G is a group with the given operation.

   3.1 $G = n\mathbb{Z} = \{nk \mid k \in \mathbb{Z}\},\ n \in \mathbb{Z}$; addition

   3.2 $G = \mathbb{R}$; $a \cdot b = a + b + 1$.

   3.3 $G = \mathbb{R}$; $a \cdot b = a + b - ab$.

   3.4 $\mathbb{Q}^+$; multiplication.

   3.5 $G = \{\sigma : \mathbb{N} \to \mathbb{N} \mid \sigma \text{ is 1-1}\}$, composition.

4. For each $n \geq 2$, the multiplication modulo n is defined on $\mathbb{Z}_n$ by

$$\overline{a} \cdot \overline{b} = \overline{ab} \qquad \text{for all } \overline{a}, \overline{b} \in \mathbb{Z}_n.$$

   4.1 Show that $(\mathbb{Z}_n, \cdot)$ is a monoid. Give an example to show that $(\mathbb{Z}_n^*, \cdot)$ may not be a group.

   4.2 Let $\mathcal{U}(n) = \{\bar{a} \in \mathbb{Z}_n \mid \text{g.c.d.}(a, n) = 1\}$. Show that $\mathcal{U}(n)$ is a group under the multiplication modulo n.

Assignment 2

1. Let G be a group and $a \in G$. Show that

   (i) The map $L_a : G \to G$ defined by $L_a(x) = ax$ is a bijection.

   (ii) The map $R_a : G \to G$ defined by $R_a(x) = xa$ is a bijection.

2. Let G be a group. For each $a \in G$, define $\phi_a : G \to G$ by

$$\phi_a(x) = axa^{-1} \text{ for all } x \in G.$$

   Show that

   (i) ϕ_a is a bijection for all $a \in G$, and

   (ii) $\phi_a \phi_b = \phi_{ab}$ for all $a, b \in G$.

3. Let a and b be elements of G. Show that

$$ab = ba \text{ if and only if } a^{-1}b^{-1} = b^{-1}a^{-1}.$$

4. Let G be a group. Show that TFAE :

   (i) G is abelian.

   (ii) $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

   (iii) $(ab)^2 = a^2b^2$ for all $a, b \in G$.

5. Let G be a group. Show that if $a^2 = e$ for all $a \in G$, then G is abelian. Give an example to show that the converse is not necessary true.

Assignment 3

1. Let H and K be a subgroups of a group G. Show that $H \cap K$ is also a subgroup of G. Given an example to show that $H \cup K$ is necessary a subgroup of G.

2. Draw the lattice of subgroups of the following groups :

   (2.1) $\mathbb{Z}_8$ (2.2) $\mathbb{Z}_{24}$ (2.3)$\mathbb{Z}_2 \times \mathbb{Z}_2$ (2.4)$\mathbb{Z}_4 \times \mathbb{Z}_{12}$.

3. Find order of each element of groups in Problem 2.

4. Determine whether the following sets are subgroups of $GL_3(\mathbb{R})$:

   (4.1) $H_1 = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}.$

   (4.2) $H_2 = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}.$

   (4.3) $H_3 = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \middle| \, abc \neq 0 \right\}.$

   (4.4) $H_4 = \left\{ \begin{bmatrix} a & b & c \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}.$

5. If H and K are subgroups of G. Show that

(5.1) $H \cup K$ is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$.

(5.2) gHg^{-1} is a subgroup of G for all $g \in G$.

(5.3) $(gHg^{-1}) \cap (gKg^{-1}) = g(H \cap K)g^{-1}$ for all $g \in G$.

6. If G is an abelian group and $n \geq 2$ is an integer. Show that the following sets are subgroups of G.

(6.1) $G^n = \{g^n \mid g \in G\}$.

(6.2) $G(n) = \{g \in G \mid g^n = e\}$.

7. If G is an abelian group, show that

$$\tau(G) = \{g \in G \mid g^k = e \text{ for some } k \in \mathbb{N}\}$$

is a subgroup of G.

8. If G_1 and G_2 are abelian groups, show that $G_1 \times G_2$ is abelian.

9. Let G be a group and A a nonempty subset of G. The *centralizer* of A in G, $C_G(A)$ and the *normalizer* of A in G, $N_G(A)$, are defined as follows :

$$C_G(A) = \left\{g \in G \mid gag^{-1} = a \text{ for all } a \in A\right\},$$
$$N_G(A) = \left\{g \in G \mid gAg^{-1} = A\right\}.$$

(9.1) Show that $C_G(A)$ and $N_G(A)$ are subgroups of G.

(9.2) Show that $Z(G) = \bigcap_{a \in G} C_G(a)$, where $C_G(a) = C_G(\{a\})$.

10. Let G be a group and $g \in G$. show that

(10.1) $\circ(g) = \circ(g^{-1})$ and $\circ(g) = \circ(xgx^{-1})$ for all $x \in G$.

(10.2) If $\circ(g) = n$, than $\circ(g^t) = \frac{n}{d}$ where $d = \text{g.c.d.}(n, t)$.

(10.3) $\circ(gh) = \circ(hg)$ for all $h \in G$.

11. Prove that a group of order 4 must be abelian.

12. Let a and b be element of a group G. If $ab = ba$ and $\circ(a)$ and $\circ(b)$ are finite show that $\circ(ab)$ is finite.

13. Show that Problem 12 falis if $ab \neq ba$.

14. Let G be a finite group and g be element of odd order of G. Show that $g = (g^2)^k$ for some k.

Assignment 4

1. In each case determine whether α is a homomorphism. If it is, determine its kernel and its image.

   1.1 $\alpha : \mathbb{Z} \to GL_2(\mathbb{Z})$ defined by $\alpha(n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$.

   1.2 $\alpha : GL_2(\mathbb{Q}) \to \mathbb{Q}^*$ defined by $\alpha(A) = detA$.

   1.3 $\alpha : \mathbb{C} \to M_2(\mathbb{R})$ defined by $\alpha(a + bi) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

   1.4 $\alpha : G \to G \times G$ defined by $\alpha(g) = (g, g)$.

   1.5 $\alpha : \mathbb{R} \to \mathbb{R}$ defined by $\alpha(x) = \lfloor x \rfloor$.

   1.6 $\alpha : \mathbb{R} \to \mathbb{R}$ defined by $\alpha(x) = 2x + 1$.

2. Given a group G, define $\phi : G \to G$ by $\phi(g) = g^{-1}$. Show that G is an abelian if and only if ϕ is a homomorphism.

3. Show that $\mathbb{Z}_2 \times \mathbb{Z}_2$ and K_4, the Klien-4 group are isomorphic.

4. Show that if σ is an isomorphism, then σ^{-1} is an isomorphism.

5. Let G be a group and $g \in G$. Define $\sigma_g : G \to G$ by

$$\sigma_g = gxg^{-1} \quad \text{for all } x \in G.$$

   Show that

   5.1 $\sigma_g \in Aut(G)$, called the **inner automorphism determined** by g.

   5.2 $Inn(G) = \{\alpha_g \mid g \in G\}$ is a subgroup of $Aut(G)$, called the **inner automorphism group** of G.

6. Let G_1 and G_2 be groups. Show that

   (i) $G_1 \times G_2 \cong G_2 \times G_1$.

   (ii) The maps $\pi_1 : G_1 \times G_2 \to G_1$ and $\pi_2 : G_1 \times G_2 \to G_2$ defined by

$$\pi_1(a_1, a_2) = a_1 \ \text{ and } \ \pi_2(a_1, a_2) = a_2$$

   are homomorphism. Find their kernels.

7. Show that

$$G = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

   is a subgroup of $G_2(\mathbb{Z})$ isomorphic to the subgroup $U_4 = \{1, -1, \imath, -\imath\}$ of $\mathbb{C}^*$.

8. Show that $\mathcal{U}(15) \cong \mathcal{U}(16)$ but $\mathcal{U}(10)$ is not isomorphic to $\mathcal{U}(12)$.

Assignment 5

1. Show that any group of prime order must be cyclic.

2. Let m and n be integers. Find a generator of the group $m\mathbb{Z} \cap m\mathbb{Z}$.

3. Show that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is a cyclic but $\mathbb{Z}_2 \times \mathbb{Z}_4$ is not.

4. Assume that G is a group that has only two subgroups $\{e\}$ and G. Show that G is a finite cyclic group of order 1 or a prime.

5. $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if and only if g.c.d.$(m, n) = 1$.

Assignment 6

1. Find the left cosets and the right cosets of $\langle(12)\rangle$ in S_3.

2. Find all the right cosets of $\{1, 11\}$ in $\mathcal{U}_{30} = \{\bar{a} \in \mathbb{Z}_{30} \mid (a, 30) = 1\}$.

3. Let $G \neq \{e\}$ be a group. Assume that G has no proper nontrivial subgroups. prove that $|G|$ is prime.

4. Give an example to show that a group of order 8 need not have an element of order 4.

5. Let G be a group of order pq where p and q are primes. Show that every proper subgroup of G is cyclic.

6. Show that if H is a subgroup of index 2 of a finite group G, then every left coset of H is also a right coset of H.

Assignment 7

1. Show that $\langle(123)\rangle$ is the only normal subgroup of S_3.

2. If H and K are normal subgroup of G, show that $H \cap K$ is a normal subgroup of G.

3. If $K \lhd H$ and $H \lhd G$, show that $aKa^{-1} \lhd H$ for all $a \in G$.

4. Give an example to show that the normality need not be transitive.

5. If $G = H \times K$, find normal subgroups H_1 and K_1 of G such that $H_1 \cong H, K_1 \cong K, H_1 \cap K_1 = \{e\}$ and $G = H_1 K_1$.

6. Let H be a subgroup of a group G. Show that

   6.1 $H \lhd N_G(H)$. ($N_G(H)$ is the largest subgroup of G in which H is normal).

   6.2 If $H \lhd K$, where K is a subgroup of G, then $K \subseteq N_G(H)$.

7. Let G be a group of order pq where p and q are distinct primes. Show that if G has a unique subgroup of order p and a unique subgroup of order q, then G is cyclic.

8. Let G be a group and $D = \{(g, g) | g \in G\}$. Show that D is a normal subgroup of G if and only if G is abelian.

9. Show that $Inn(G)$ is normal in $Aut(G)$.

10. Let $G = S_3$ and $H = \langle(123)\rangle$. Tabulate the operation of G/H.

11. Let N be a normal subgroup of prime index in a group G. Show that G/N is cyclic.

12. Let a be an element of order 4 in a group G of order 8. Let $b \in G \smallsetminus \langle a \rangle$. Show that

    12.1 $b^2 \in \langle a \rangle$.

    12.2 If $\circ(b) = 4$, then $b^2 = a^2$.

13. Let G be a group. If $G/Z(G)$ is cyclic, show that G is abelian.

14. Show that if a finite group G has exactly one subgroup H of a given order, then H is a normal subgroup of G.

15. Let N be a normal subgroup of G and let $m = [G : N]$. Show that $a^m \in N$ for every $a \in G$.

16. Let $H \lhd G$ and $H' \lhd G'$. Let $\phi : G \to G'$ be a homomorphism. Show that ϕ induces a homomorphism $\phi_a : G/H \to G'/H'$ if $\phi[H] \subseteq H'$.

Assignment 8

1. Calculate all conjugacy classes of the following groups :

   1.1 Q_8 1.2 K_4 1.3 A_4 1.4 S_4.

2. Decribe the conjugacy classes of an abelian group.

3. Show that ab and ba are conjugate in any group.

4. If a subgroup H of G is a union of conjugacy classes in G, show that $H \lhd G$.

5. Show that, upto isomorphism, there are exactly two groups of order 4.

Assignment 9

1. Determine whether groups in each problem are isomorphic.

   1.1 $\mathbb{Q}_8$ and $\mathbb{Z}_8$

   1.2 $\mathbb{Z}_4$ and K_4

   1.3 S_3 and $\mathbb{Z}_6$

   1.4 $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_6$.

2. Let G be a group. Show that $G/Z(G) \cong Inn(G)$.

3. Show that $SL_n(\mathbb{Q})$ is a normal subgroup of $GL_n(\mathbb{Q})$.

4. Let M and N be normal subgroups of G such that $G = MN$. Prove that

$$G/(M \cap N) \cong G/M \times G/N.$$

5. Let $S = \{z \in \mathbb{C}^* \mid |z| = 1\}$. Show that

   5.1 S is a subgroup of the multiplicative group of nonzero complex numbers $\mathbb{C}^*$.

   5.2 $\mathbb{R}/\mathbb{Z} \cong S$ where $\mathbb{R}$ is the additive group of real numbers.

Assignment 10

1. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 1 & 6 & 4 \end{pmatrix}$.

   (i) Compute α^{-1}, $\alpha\beta$, $\beta\alpha$ and $\alpha\beta^{-1}$.

   (ii) Write α and β in cycle form and as product of transpositions.

   (ii) Find orders of α, α^{-1} and $\alpha\beta$

2. Write the lattice of subgroups of A_4.

3. Prove that the subgroup of order 4 in A_4 is normal and is isomorphic to K_4, the Klien 4-group.

4. Prove that $\langle (13), (1234) \rangle$ is a proper subgroup of S_4.

5. Prove that σ^2 is an even permutation for every permutation σ.

6. Show that $sgn(\sigma) = sgn(\sigma^{-1})$ for all $\sigma \in S_n$.

7. Show that $\alpha^{-1}\beta^{-1}\alpha\beta$ is an even permutation for all $\alpha, \beta \in S_n$.

8. Show that A_5 contains an element of order 6.

9. Is the product of two odd permutation an even or an odd permutation.

10. Determine whether the following permutations are even or odd.

    (i) (237) (ii) (12)(34)(153) (iii) (1234)(5321)

11. Do the odd permutations in S_n from a group ? justify your answer.

12. Show that A_n is generated by the set of 3-cycles.

13. Show that $S_n = \langle (12), (12 \dots n) \rangle$ for all $n \geq 2$.

14. Show that any two elements of S_n are conjugate in S_n if and only if they have the same cycle type.

15. Find all conjugacy classes of S_4.

16. Find all left cosets and right cosets of $H = \{(1), (12)(34), (13)(24), (14)(23)\}$ in A_4.